

Linear Independence

Eg: (HW#3)

$\text{Span} \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$ is the plane $-13b_1 - 5b_2 + b_3 = 0$

Why a plane and not \mathbb{R}^3 ? The vectors are coplanar: one is in the span of the others.

$$\frac{5}{2} \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \quad [\text{demo}]$$

Any two non-collinear vectors span a plane:

$$\text{Span} \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \right\}$$

This reduces the number of parameters needed to describe this set: not scalar multiples

$$x_1 \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \quad \text{vs.} \quad x_1 \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$$

Moreover, the expression with 2 parameters is unique, but with 3 parameters it is redundant:

$$1 \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} - 1 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = 6 \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} - 7 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$$

$$\text{but } \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \quad \text{only for}$$

[demo] $x_1 = 1, x_2 = -1$

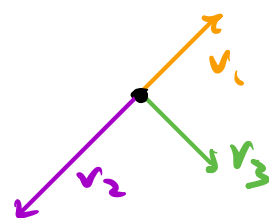
We want to formalize this notion that there are "too many" vectors spanning this subspace by saying one is **in the span of the others**.

In the above example, each vector is in the span of the other 2, but this need not be the case.

Eg: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Here $v_2 = -2v_1 + 0v_3$

but $v_3 \notin \text{Span}\{v_1, v_2\}$



We want a condition that means **some** vector is in the span of the others. Answer: rewrite as a **homogeneous vector equation**:

Eg: $\frac{5}{2} \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} = 0$ $-2v_1 - v_2 + 0v_3 = 0$

Def: A list of vectors $\{v_1, \dots, v_n\}$ is **linearly dependent (LD)** if the vector equation

$$x_1 v_1 + \dots + x_n v_n = 0$$

has a **nontrivial** solution. Such a solution is called a **linear relation** among $\{v_1, \dots, v_n\}$

LD means the system $x_1v_1 + \dots + x_nv_n = 0$ has a free variable.

The above Eg gives linear relations \rightarrow the sets $\left\{\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$ are LD.

NB: If $x_1v_1 + \dots + x_nv_n = 0$ and $x_i \neq 0$ then
$$v_i = -\frac{1}{x_i} (x_1v_1 + \dots + x_{i-1}v_{i-1} + x_{i+1}v_{i+1} + \dots + x_nv_n)$$
 so v_i is in the span of the others.

LD means some vector is in the span of the others: $x_1v_1 + \dots + x_nv_n = 0$ and $x_i \neq 0$ implies $v_i \in \text{Span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$

Def: A list of vectors $\{v_1, \dots, v_n\}$ is linearly independent (LI) if it is not linearly dependent: ie, if the vector equation
$$x_1v_1 + \dots + x_nv_n = 0$$
 has only the trivial solution. ie if $x_1v_1 + \dots + x_nv_n = 0$ then $x_1 = \dots = x_n = 0$.

LI means no vector is in the span of the others.

Roughly, vectors v_1, \dots, v_n are LI if their span is **as large as it can be**. Every time you add a vector, the span gets **bigger**!

Eg: Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ LI or LD?

In other words, does the vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

have a nontrivial solution? **free \Rightarrow LD**

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{PF}} \begin{matrix} x_1 = -x_3 \\ x_2 = 2x_3 \end{matrix}$$

Take $x_3 = 1 \rightarrow$ **linear relation**

$$-\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

So they're **LD** [demo]

Eg: Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ LI or LD?

In other words, does the vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

have a nontrivial solution?

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -22 \\ 0 & 0 & 32 \end{bmatrix}$$

No free variables \Rightarrow only the trivial solution
 \Rightarrow these vectors are LI [demo]

Fact: If $\{v_1, \dots, v_n\}$ is LI and
 $b \in \text{Span}\{v_1, \dots, v_n\}$ then there are unique
weights x_1, \dots, x_n such that

$$b = x_1 v_1 + \dots + x_n v_n$$

In other words, this is not a redundant
parameterization of $\text{Span}\{v_1, \dots, v_n\}$


Proof: Let A be the matrix with cols v_1, \dots, v_n

$$\text{so } Ax = b \equiv x_1 v_1 + \dots + x_n v_n = b$$

so $Ax = 0$ has only the trivial solution
ie $\text{Nul}(A) = \{0\}$.

$Ax = b$ is consistent because

$$b \in \text{Span}\{v_1, \dots, v_n\} = \text{Col}(A)$$

so $\{\text{sols of } Ax = b\} = (\text{particular soln}) + \text{Nul}(A)$
 \hookrightarrow only one solution 

Linguistic note: LI, LD are adjectives that apply
to a set of vectors.

Bad: "A is LI" "v₁ is LD on v₂ and v₃"

Good: "A has LI columns" " $\{v_1, v_2, v_3\}$ is LD"

Eg: • $\{v\}$ is LI $\Leftrightarrow v \neq 0$

- Any set containing the 0 vector is LD:
if $v_i = 0$ then

$$0 = 1 \cdot v_i + 0 \cdot v_2 + \dots + v_n$$

is a linear relation.

- Suppose $\{v, w\}$ is LD. So there exist $(a, b) \neq (0, 0)$ such that $av + bw = 0$.

$$\left. \begin{array}{l} a \neq 0 \leadsto v = -\frac{b}{a}w \\ b \neq 0 \leadsto w = -\frac{a}{b}v \end{array} \right\} v, w \text{ are collinear.}$$

$\{v, w\}$ is LD $\Leftrightarrow v, w$ are collinear.

- Similar, $\{u, v, w\}$ is LD $\Leftrightarrow u, v, w$ are coplanar, and so on.

- If $r > n$ then r vectors in \mathbb{R}^n are LD:
the matrix

$$\begin{bmatrix} | & & | \\ v_1 & \dots & v_r \\ | & & | \end{bmatrix}$$

is wide, so it has a free variable.

eg. 3 vectors in \mathbb{R}^2 are automatically LD.

[demo]

Basis and Dimension

A basis of a subspace is a **minimal** set of vectors needed to span (parameterize) that subspace.

Def: A set of vectors $\{v_1, \dots, v_n\}$ is a **basis** for a subspace V if:

(1) $V = \text{Span}\{v_1, \dots, v_n\}$

(2) $\{v_1, \dots, v_n\}$ is **linearly independent**

The **dimension** of V is the number of vectors in **any** basis. (Fact: all bases have the same size!)

Notation: $\dim(V)$

Spans means you get a **parameterization** of V :

$$b \in V \Rightarrow b = x_1 v_1 + \dots + x_n v_n$$

LI means this parameterization is **unique**.

Rephrase: A **spanning** set for V is a **basis** if it is **linearly independent**.

Eg: $V = \text{Span} \left\{ \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$

A basis is $\left\{ \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$. (or $\left\{ \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$)

(1) **Spans**: because $\begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}\right\}$

(2) **LI**: because not collinear.

So $\dim(V) = 2$ (a plane) ✓

Eg: $\{0\} = \text{Span}\{\} \Rightarrow \dim\{0\} = 0$ ✓

Eg: A **line** L is spanned by one vector
 $\Rightarrow \dim(L) = 1$.

In general:

- A **point** has dimension **0**
 - A **line** has dimension **1**
 - A **plane** has dimension **2**
- etc.

Eg: What is a basis for \mathbb{R}^n ?

The **unit coordinate vectors** $e_1 \rightarrow e_n$.

$$n=3: \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$x_1 e_1 + x_2 e_2 + x_3 e_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(1) **Spans**: every vector has this form.

(2) **LI**: if this = 0 then $x_1 = x_2 = x_3 = 0$ ✓

So $\dim(\mathbb{R}^n) = n$ ✓

NB: \mathbb{R}^n has many bases.

eg. \mathbb{R}^2 is spanned by any pair of noncollinear vectors: $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$; $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$; $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}, \dots$

In fact, any nonzero subspace has infinitely many bases! Parameterizations are not unique!

NB: Be careful to distinguish between a basis for a subspace and the subspace it spans.

$$V = \text{Span} \left\{ \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$$

↑ This is a subspace. It is a plane. It has ∞ vectors in it.

$$\left\{ \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\} \leftarrow$$

This is a basis for V . It has 2 vectors in it. It is a finite list of data that describes V .

Bases for $\text{Col}(A)$ & $\text{Nul}(A)$

Remember, if someone hands you a subspace, you want to write it as a column space or a null space so you can do computations, like find a basis.

Thm: The pivot columns of A form a basis of $\text{Col}(A)$.

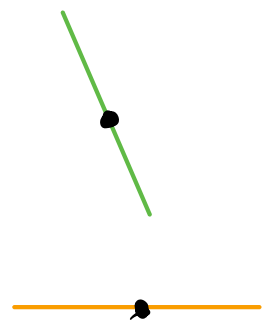
$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{basis: } \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

↑ pivot column

NB: Take the pivot columns of the original matrix, Not the RREF. Doing row ops changes the column space!

$$\text{Col} \begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

$$\text{Col} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$



Proof: Let R be the RREF of A .

$$A = \begin{bmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the pivot columns are v_1, v_2, v_4 .

Note: $Ax=0 \iff Rx=0$ (same solution set)

(1) Spans: $\begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow 0 = -3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 6 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow R \begin{bmatrix} -3 \\ -2 \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow A \begin{bmatrix} -3 \\ -2 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow v_3 = 3v_1 + 2v_2$$

A and R have the same col relations!

Similarly, $\begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow v_5 = 4v_1 + 6v_2 - v_4$$

Any vector in $\text{Col}(A)$ has the form

$$\begin{aligned}
 v &= x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 + x_5 v_5 \\
 &= x_1 v_1 + x_2 v_2 + x_3 (3v_1 + 2v_2) + x_4 v_4 + x_5 (4v_1 + 6v_2 - v_4) \\
 &= (x_1 + 3x_3 + 4x_5) v_1 + (x_2 + 2x_3 + 6x_5) v_2 + (x_4 - x_5) v_4
 \end{aligned}$$

which is in $\text{Span} \{v_1, v_2, v_4\}$.

(2) **LI**: If $x_1 v_1 + x_2 v_2 + x_4 v_4 = 0$ then

$$A \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0 \Rightarrow R \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 = x_4 = 0 \quad \checkmark$$

Consequence: The number of vectors in a basis for $\text{Col}(A)$ is equal to the number of pivots of A .

$$\text{rank}(A) = \dim \text{Col}(A)$$

Eg: Find a basis for $\text{Span} \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$

Step 0: Rewrite as $\text{Col} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

Now find pivot columns:

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 2 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Basis: } \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \right\}$$

2 pivots \leadsto Span is a plane.

Thm: The vectors attached to the free variables in the parametrized vector form of the solution set of $Ax=0$ form a basis for $\text{Nul}(A)$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{PVE}} x = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{basis: } \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Proof:

(1) Spans: Every solution $= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ ✓

(2) **LI:** Think about it in parametric form:

$$0 = x_1 = -2x_2 + x_4$$

$$0 = x_2 = x_2$$

$$0 = x_3 = -x_4$$

$$0 = x_4 = x_4$$

↑ trivial equations

$$\Rightarrow x_2 = x_4 = 0$$



Consequence:

$$\dim \text{Nul}(A) = \# \text{free vars} = \# \text{cols} - \text{rank}$$

NB: This is consistent with our provisional definition of the dimension of a solution set as the number of free variables.