Math 218D Problem Session: Week 11

November 11, 2022

1. Matrices with complex eigenvalues

Consider the matrices $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) Compute the eigenvalues of A and B. Write each eigenvalue in polar coordinates $z = re^{i\theta}$.

b) Compute the eigenvectors of A and B.

2. **Some quick matrix exponentials**

Compute the matrix exponential e^A of:

(1) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$,

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,

(3) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(4) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

(5) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

3. A differential equation

Consider the system of differential equations

$$x'(t) = 3x(t) + 2y(t)$$
$$y'(t) = 4x(t) - 4y(t)$$

a) Write this as a matrix differential equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

What is the matrix A?

b) For this matrix A, find the eigenvalues λ_1 and λ_2, as well as the eigenvectors w_1 and w_2.

c) Every solution is of the form $(x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. If you want the solution to have initial value $(x(0), y(0)) = (1, 1)$, which scalars a_1 and a_2 should you choose?

d) Plug the solution with initial value $(x(0), y(0)) = (1, 1)$ to the differential equation, and confirm that it is a solution.

e) For the solution you found in **c)**, compute $(x(1), y(1))$.

4. A complex ODE

Consider the system of differential equations

$$x'(t) = x(t) - y(t),$$
$$y'(t) = x(t) + y(t).$$

a) Write this as a matrix differential equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$.

b) Compute the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix A.

c) Compute the real and imaginary parts of the "eigenvector solution" $(x(t), y(t)) = e^{\lambda_1 t} v_1$. This gives you two different *real* solutions to the differential equation.

d) Find the solution $(x(t), y(t))$ with initial value $(x(0), y(0)) = (1, 0)$.

5. The dynamics of a diagonal matrix

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

a) For each of the following vectors, plot v, Av, and A^2v:

(1) $v = (1,0)$

(2) $v = (0,1)$

(3) $v = (1,1)$

b) For each of the same vectors, sketch the shape you get by connecting the dots between the points $\ldots, A^{-2}v,\ A^{-1}v,\ v,\ Av,\ A^2v, \ldots$.

c) For the vector $v = (1,1)$, what direction is the vector $A^n v$ approximately pointing when n is very large? In other words, what unit vector does $\frac{A^n v}{||A^n v||}$ approximate when n is very large?

d) For the vector $v = (1,1)$, what direction is $A^{-n}v$ approximately pointing when n is very large?