

Math 218D Problem Session: Week 12

November 18, 2022

1. Shape of quadratic forms

For each of the following quadratic forms:

(1) Plot the equation $q(x, y) = 1$ using a computer, and describe the shape (for example, for **a**) you should get an ellipse in \mathbf{R}^2 , not an elliptic paraboloid in \mathbf{R}^3).

(2) Find the 2×2 symmetric matrix $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that

$$q(x, y) = \begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

(3) Recall that a symmetric matrix is **positive-definite** if all of its eigenvalues are positive. Test if the symmetric matrix S is positive-definite or not using the **pivot test**: Put S into REF without doing row-swaps or scaling. (If you need to do a row-swap, the matrix is not positive-definite.) If the diagonal entries of the REF are all positive, then S is positive-definite.

(4) What does the positive-definiteness of S have to do with the shape from (1)? You may need to do many examples until you see the pattern.

a) $q(x, y) = 2x^2 + 3y^2$

b) $q(x, y) = x^2 - 5y^2$

c) $q(x, y) = y^2$

d) $q(x, y) = -3x^2 - 2y^2$

e) $q(x, y) = x^2 + 3xy + y^2$

f) $q(x, y) = 2x^2 + 4xy + y^2$

g) $q(x, y) = x^2 - 4xy + 5y^2$

Our final two quadratic forms are in 3 variables: this means that S is a 3×3

matrix, and $q(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} S \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

h) $q(x, y, z) = x^2 + y^2 + z^2 + xy + yz + xz$

i) $q(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$

2. Diagonalizing quadratic forms

Consider the quadratic form

$$q(x, y) = \frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2.$$

- a) What is the symmetric matrix S so that $q(x, y) = (x \ y)S \begin{pmatrix} x \\ y \end{pmatrix}$?
- b) Find the symmetric diagonalization, $S = QDQ^T$, where the matrix Q is orthonormal. The columns of Q are orthonormal eigenvectors v_1 and v_2 , with eigenvalues λ_1 and λ_2 .
- c) The quadratic form for the diagonal matrix D is $(x_0 \ y_0)D \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda_1^2 x_0^2 + \lambda_2^2 y_0^2$. Plot $q(x, y) = 1$ and $\lambda_1 x_0^2 + \lambda_2 y_0^2 = 1$. What is the geometric relationship between these shapes?
- d) Confirm that $q(x, y) = \lambda_1 x_0^2 + \lambda_2 y_0^2$, where we relate the variables x_0 and y_0 to the variables x and y using $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = Q^T \begin{pmatrix} x \\ y \end{pmatrix}$.
- e) Using d), explain why the equation $q(x, y) = 1$ describes an ellipse. How does this relate to the **eigenvalue test** for positive-definite matrices?
- f) Using d), explain why the function $q(x, y)$ is always positive (unless $x = y = 0$). How does this relate to the **positive-energy test** for positive-definite matrices?
- g) Consider the **constrained optimization** problem: what is the maximum value of the function $q(x, y)$ on the unit circle $x^2 + y^2 = 1$? at what point (x, y) on the unit circle does q achieve the maximum?
Hint: First answer these questions for the quadratic form $\lambda_1 x_0^2 + \lambda_2 y_0^2$.

3. LDL^T decomposition

Find the LDL^T decomposition of the following positive-definite symmetric matrices, by:

(1) Computing the $A = LU$ decomposition.

(2) Setting $D =$ the diagonal matrix with same diagonal entries as U .

a) $S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

b) $S = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}$

4. Relation to the quadratic formula

For 2×2 symmetric matrices $S = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$, there is an easy test for positive-definiteness, the **discriminant test**:

S is positive-definite if and only if both $a > 0$ and $b^2 - 4ac < 0$.

Let's verify this test in two ways, by relating it to other tests.

a) Method one: Relate the discriminant test to the **determinant test**: S is positive-

definite if and only if $\det(a) > 0$ and $\det\left(\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}\right) > 0$.

b) Method two:

(1) Show that the quadratic form $q(x, y) = (x, y)^T S(x, y)$ equals

$$q(x, y) = ax^2 + bxy + cy^2$$

and factors into

$$q(x, y) = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2}y\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2}y\right).$$

(2) If $b^2 - 4ac < 0$, explain why $q(x, y) \neq 0$ for all real numbers x, y (not both zero).

This means that either $q(x, y) > 0$ for all $(x, y) \neq (0, 0)$ or $q(x, y) < 0$ for all $(x, y) \neq (0, 0)$.

(3) If both $a > 0$ and $b^2 - 4ac < 0$, explain why $q(x, y) > 0$ for all real numbers x, y (not both zero).

Hint: If $a > 0$, can you find a point (x, y) where $q(x, y) > 0$?

This show that **if S satisfies the discriminant test, it satisfies the energy test.**