Math 218D Problem Session: Week 4

Answer Key

1. Subspaces?

Decide if each of the following sets of vectors is or is not a subspace. If so, express the subset as a null space or a column space of a matrix. If not, give a counterexample to one of the subspace axioms.

a) $$\{(x,y,z)\in\mathbf{R}^3 \;:\; x+y=1-z\}$$

b) $$\{(x,y)\in\mathbf{R}^2 \;:\; x-2y=0\}$$

c) $$\left\{v\in\mathbf{R}^3 \;:\; Av=\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \qquad (A \text{ a } 3\times 3 \text{ matrix})$$

d) $$\left\{(x,y)\in\mathbf{R}^2 \;:\; \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ -1 & -2 & -3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\end{pmatrix}\right\}$$

e) $$\{(x,y)\in\mathbf{R}^2 \;:\; x^2+y^2=1\}$$

f) $$\{(x,y)\in\mathbf{R}^2 \;:\; x^2+2xy+y^2=0\}$$

Solution.

a) Not a subspace, since it doesn't contain $(0,0,0)$.

b) This is $\mathrm{Nul}\begin{pmatrix}1 & -2\end{pmatrix}$.

c) Not a subspace, since it doesn't contain $(0,0,0)$.

d) We take transposes of both sides of the equation:

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ -1 & -2 & -3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{\text{becomes}} \begin{pmatrix}1 & -1\\ 2 & -2\\ 3 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Hence this subspace is the null space of the matrix

$$\begin{pmatrix}1 & -1\\ 2 & -2\\ 3 & -3\end{pmatrix}.$$

e) Not a subspace, since it doesn't contain $(0,0,0)$.

f) This one is tricky. Note that $x^2+2xy+y^2=(x+y)^2$ is equal to zero if and only if $x+y=0$, so

$$\{(x,y)\in\mathbf{R}^2 : x^2+2xy+y^2=0\}=\{(x,y)\in\mathbf{R}^2 : x+y=0\}=\mathrm{Nul}\begin{pmatrix}1 & 1\end{pmatrix}.$$

The four *fundamental subspaces* associated to a matrix A are

$$\mathrm{Nul}(A), \quad \mathrm{Col}(A), \quad \mathrm{Nul}(A^T), \quad \mathrm{Col}(A^T).$$

2. The fundamental subspaces I

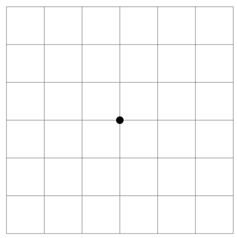
Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

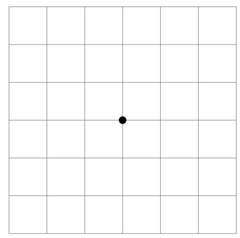
a) Find a spanning set for each of the four fundamental subspaces of this matrix.

b) Draw each of the fundamental subspaces:

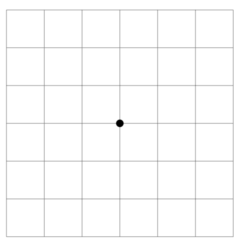
Nul(A)



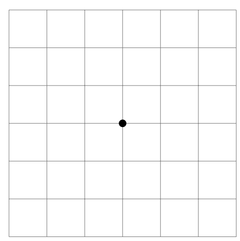
Col(A)



Nul(A^T)



Col(A^T)



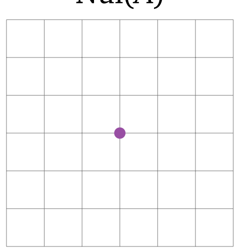
c) Compute $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T))$.

Solution.

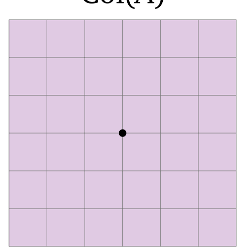
a) The subspaces Nul(A) and Nul(A^T) are points, while Col(A) and Col(A^T) are all of $\mathbf{R}^2$. Hence $\{\}$ is a spanning set for Nul(A) and Nul(A^T), and any pair of noncollinear vectors form a spanning set for Col(A) and Col(A^T).

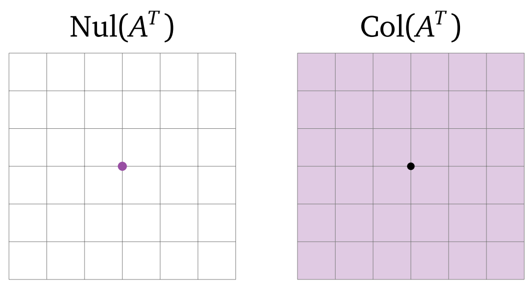
b) We have to draw a point and a plane in each case:

Nul(A)



Col(A)





c) $\dim(\text{Nul}(A)) + \dim(\text{Col}(A^T)) = 2$

3. The fundamental subspaces II

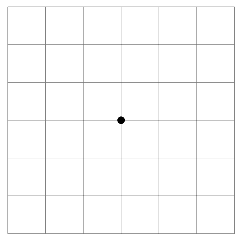
Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}.$$

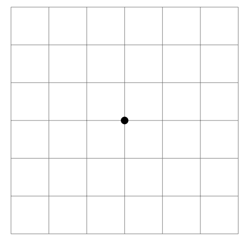
a) Find a spanning set for each of the four fundamental subspaces of the matrix.

b) Draw each of the fundamental subspaces:

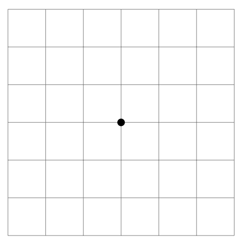
Nul(A)



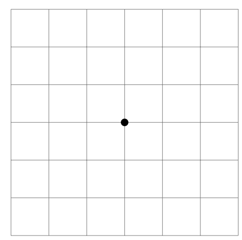
Col(A)



Nul(A^T)



Col(A^T)



c) Compute $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T))$.

d) Describe the geometric relationship between Nul(A) and Col(A^T) and between Col(A) and Nul(A^T).

Solution.

a) One possible spanning set for each subspace is

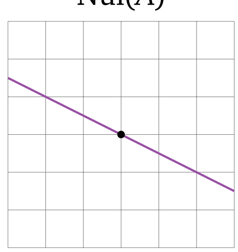
$$\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\} \qquad \mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

$$\mathrm{Nul}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \qquad \mathrm{Col}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}.$$

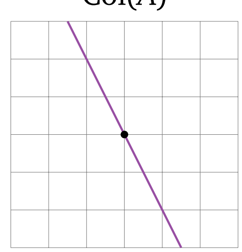
Other answers are possible—you can scale each vector by a nonzero number, for instance.

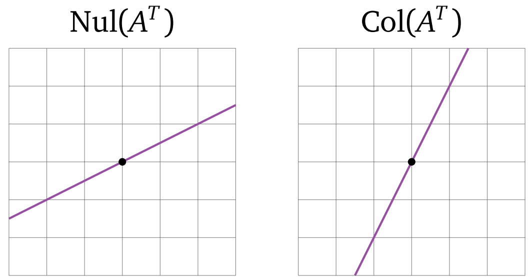
b) We have to draw four lines:

Nul(A)



Col(A)





c) $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T)) = 2$.

d) The lines Nul(A) and Col(A^T) are perpendicular. The lines Col(A) and Nul(A^T) are perpendicular.

4. The fundamental subspaces III

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Is $\mathrm{Col}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

b) Is $\mathrm{Nul}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

c) Is $\mathrm{Col}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

d) Is $\mathrm{Nul}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

e) Two of the four subspaces are contained in $\mathbf{R}^2$. Draw these two subspaces, and describe the geometric relationship between them.

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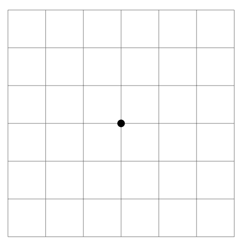

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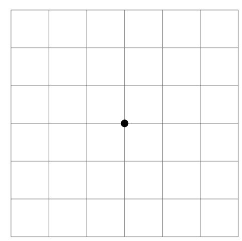


f) Two of the four subspaces are contained in $\mathbf{R}^3$. For this matrix, one is a line and the other is a plane. Determine which is which, and find bases for both of these subspaces.

g) Find an implicit equation $a_1x + a_2y + a_3z = 0$ for the plane in **f)**.

h) What can you observe about the relationship between the answers to **f)** and **g)**? What does this mean geometrically?

Solution.

a) $\mathrm{Col}(A^T)$ is a subspace of $\mathbf{R}^3$.

b) $\mathrm{Nul}(A)$ is a subspace of $\mathbf{R}^3$.

c) $\mathrm{Col}(A)$ is a subspace of $\mathbf{R}^2$.

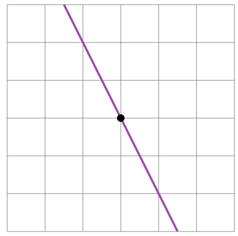
d) $\mathrm{Nul}(A^T)$ is a subspace of $\mathbf{R}^2$.

e) We compute

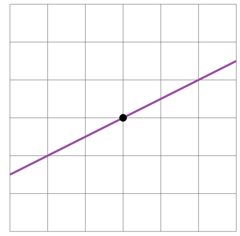
$$\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\} \quad \text{and} \quad \mathrm{Nul}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}.$$

These lines are perpendicular:

Col(A)



Nul(A^T)



f) The subspace Col(A^T) is spanned by the vectors $(1,-1,2)$ and $(-2,2,-4)$, but these are scalar multiples of each other, so Col(A^T) is a line; either vector forms a basis. The parametric vector form of the solution set of $Ax = 0$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

so $(1,1,0)$ and $(-2,0,1)$ form a basis for Nul(A).

g) The matrix equation $Ax = 0$ translates into the system of equations

$$\begin{aligned} x - y + 2z &= 0 \\ -2x + 2y - 4z &= 0. \end{aligned}$$

These equations are scalar multiples of each other, so the plane Nul(A) is determined by either of these equations. For concreteness, we express Nul(A) as the solution set of the first equation:

$$x - y + 2z = 0.$$

h) The coefficients of the equation above are $(1,-1,2)$. This vector is a basis for Col(A^T) (you may have gotten a scalar multiple of the this vector in **f)**). This means that every vector in the plane is perpendicular to the vector $(1,-1,2)$, i.e., that the plane has *normal vector* $(1,-1,2)$. In other words, *the null space is orthogonal to the row space*. We will discuss the orthogonality of the fundamental subspaces in more detail next week.

5. Linear (in)dependence I

a) Are the vectors $\binom{1}{0}$, $\binom{1}{-1}$ linearly independent? If not, write down a linear relation.

b) Are the vectors $\binom{1}{1}$, $\binom{1}{-1}$, $\binom{3}{2}$ linearly independent? If not, write down a linear relation.

c) What is the dimension of

$$\mathrm{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}?$$

d) Consider two linearly independent vectors $u, v \in \mathbf{R}^n$. Show that the two vectors $u+v$, $u-v$ are linearly independent.

e) Consider three vectors $u, v, w \in \mathbf{R}^n$. Show that the three vectors $u+v$, $u+2v-w$, $v-w$ are linearly *dependent*.

f) Show that the vectors

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix}, \begin{pmatrix}2\\3\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

are linearly dependent by writing down a linear relation among them.

Solution.

a) Since neither vector is a scalar multiple of the other, the two vectors are linearly independent.

b) Any three vectors in $\mathbf{R}^2$ must be linearly dependent. To find a linear relation, we solve the vector equation

$$x_1\begin{pmatrix}1\\1\end{pmatrix} + x_2\begin{pmatrix}1\\-1\end{pmatrix} + x_3\begin{pmatrix}3\\2\end{pmatrix} = 0$$

by solving the matrix equation $Ax = 0$, where

$$A = \begin{pmatrix}1 & 1 & 3\\1 & -1 & 2\end{pmatrix}.$$

The parametric vector form of the solution set of $Ax = 0$ is

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_3\begin{pmatrix}-5/2\\-1/2\\1\end{pmatrix}.$$

Choosing $x_3 = 2$ gives the solution $(x_1, x_2, x_3) = (-5, -1, 2)$, so

$$-5\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} + 2\begin{pmatrix}3\\2\end{pmatrix} = 0$$

is a linear relation.

c) The dimension of the span is the same as the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix},$$

since the rank of a matrix equals the dimension of its column space. We put A in row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There are 3 pivots, so its rank is 3, and hence the span has dimension 3.

d) Consider any two scalars a, b such that

$$a(u+v) + b(u-v) = 0.$$

We need to show that both of these scalars are in fact equal to 0. We rewrite the equation above as

$$(a+b)u + (a-b)v = 0.$$

Since u and v are linearly independent, this implies that $a+b=0$ and $a-b=0$. It follows that $a=b$ and $a=-b$, which implies that $a=b=0$. This means that $u+v$ and $u-v$ are linearly independent.

e) The vectors $u+v$, $u+2v-w$, $v-w$ are linearly dependent because

$$(u+v) + (v-w) - (u+2v-w) = 0.$$

f) We solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

by solving the matrix equation $Ax = 0$ for

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & -4 & 1 & 1 \end{pmatrix}.$$

The parametric vector form of the solution set of $Ax = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -1/4 \\ 1/4 \\ 1/4 \\ 1 \end{pmatrix}.$$

Choosing $x_4 = 4$ gives the solution $(x_1, x_2, x_3, x_4) = (-1, 1, 1, 4)$, so

$$-\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix} + \begin{pmatrix}2\\3\\0\\1\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

is a linear relation.

6. Linear (in)dependence II

For each of the following statements, find examples of a 2×2 matrix A and vectors $u, v \in \mathbf{R}^2$ such that the statement holds. If it is impossible to do so, explain why.

a) u, v are linearly independent, but Au, Av are linearly dependent.

b) A is invertible and $\{u, v\}$ are linearly independent, but $\{Au, Av\}$ is linearly dependent.

c) u, v are linearly dependent, but Au, Av are linearly independent.

d) u, v are linearly dependent, but Au, v are linearly independent.

Solution.

a) Take $u = \binom{1}{0}$, $v = \binom{0}{1}$, and $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

b) This is impossible: if $\{u, v\}$ is linearly independent then $\{Au, Av\}$ is also linearly independent. To see this, suppose that $aAu + bAv = 0$. We can rewrite this as $A(au + bv) = 0$. Multiplying both sides by A^{-1} gives $au + bv = A^{-1}0 = 0$. This implies $a = b = 0$ because $\{u, v\}$ is linearly independent. It follows that $\{Au, Av\}$ is also linearly independent.

c) This is impossible: if $au + bv = 0$ is a linear relation between u, v then

$$0 = A(au + bv) = aAu + bAv$$

is a linear relation between Au, Av.

d) Take $u = v = \binom{1}{0}$ and $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.