Math 218D Problem Session: Week 4

September 23, 2022

1. Subspaces?

Decide if each of the following sets of vectors is or is not a subspace. If so, express the subset as a null space or a column space of a matrix. If not, give a counterexample to one of the subspace axioms.

a) $$\{(x,y,z)\in\mathbf{R}^3 \;:\; x+y=1-z\}$$

b) $$\{(x,y)\in\mathbf{R}^2 \;:\; x-2y=0\}$$

c) $$\left\{v\in\mathbf{R}^3 \;:\; Av=\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \qquad (A \text{ a } 3\times 3 \text{ matrix})$$

d) $$\left\{(x,y)\in\mathbf{R}^2 \;:\; \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ -1 & -2 & -3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\end{pmatrix}\right\}$$

e) $$\{(x,y)\in\mathbf{R}^2 \;:\; x^2+y^2=1\}$$

f) $$\{(x,y)\in\mathbf{R}^2 \;:\; x^2+2xy+y^2=0\}$$

The four *fundamental subspaces* associated to a matrix A are

$$\mathrm{Nul}(A), \quad \mathrm{Col}(A), \quad \mathrm{Nul}(A^T), \quad \mathrm{Col}(A^T).$$

2. The fundamental subspaces I

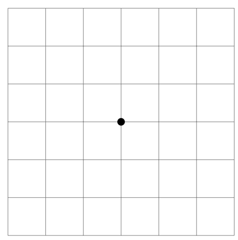
Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

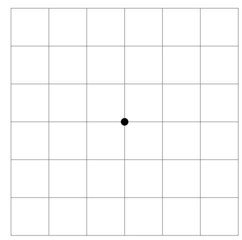
a) Find a spanning set for each of the four fundamental subspaces of this matrix.

b) Draw each of the fundamental subspaces:

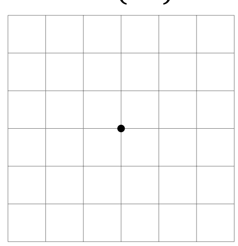
$\mathrm{Nul}(A)$



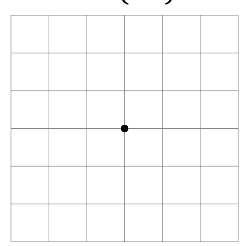
$\mathrm{Col}(A)$



$\mathrm{Nul}(A^T)$



$\mathrm{Col}(A^T)$



c) Compute $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T))$.

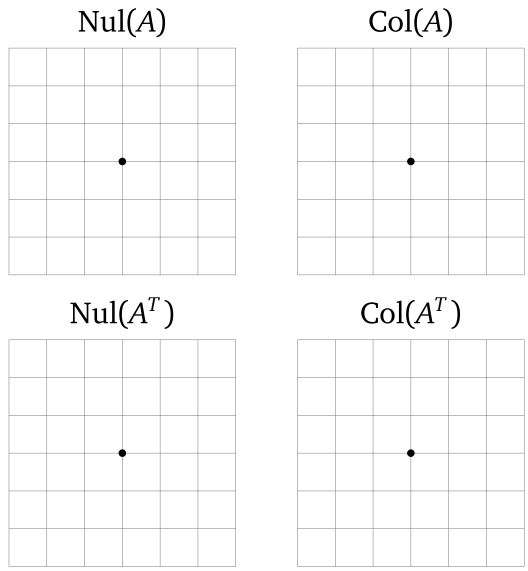
3. The fundamental subspaces II

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}.$$

a) Find a spanning set for each of the four fundamental subspaces of the matrix.

b) Draw each of the fundamental subspaces:



c) Compute $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T))$.

d) Describe the geometric relationship between $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ and between $\mathrm{Col}(A)$ and $\mathrm{Nul}(A^T)$.

4. The fundamental subspaces III

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Is $\mathrm{Col}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

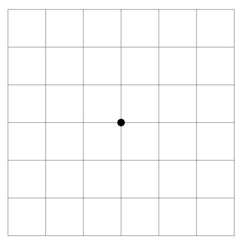
b) Is $\mathrm{Nul}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

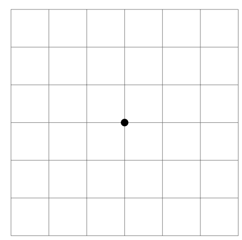
c) Is $\mathrm{Col}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

d) Is $\mathrm{Nul}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

e) Two of the four subspaces are contained in $\mathbf{R}^2$. Draw these two subspaces, and describe the geometric relationship between them.

____? ____?





f) Two of the four subspaces are contained in $\mathbf{R}^3$. For this matrix, one is a line and the other is a plane. Determine which is which, and find bases for both of these subspaces.

g) Find an implicit equation $a_1x + a_2y + a_3z = 0$ for the plane in **f)**.

h) What can you observe about the relationship between the answers to **f)** and **g)**? What does this mean geometrically?

5. Linear (in)dependence I

a) Are the vectors $\binom{1}{0}$, $\binom{1}{-1}$ linearly independent? If not, write down a linear relation.

b) Are the vectors $\binom{1}{1}$, $\binom{1}{-1}$, $\binom{3}{2}$ linearly independent? If not, write down a linear relation.

c) What is the dimension of

$$\operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}?$$

d) Consider two linearly independent vectors $u, v \in \mathbf{R}^n$. Show that the two vectors $u + v$, $u - v$ are linearly independent.

e) Consider three vectors $u, v, w \in \mathbf{R}^n$. Show that the three vectors $u+v$, $u+2v-w$, $v - w$ are linearly *dependent*.

f) Show that the vectors

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix}, \begin{pmatrix}2\\3\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

are linearly dependent by writing down a linear relation among them.

6. Linear (in)dependence II

For each of the following statements, find examples of a 2×2 matrix A and vectors $u, v \in \mathbf{R}^2$ such that the statement holds. If it is impossible to do so, explain why.

a) u, v are linearly independent, but Au, Av are linearly dependent.

b) A is invertible and $\{u, v\}$ are linearly independent, but $\{Au, Av\}$ is linearly dependent.

c) u, v are linearly dependent, but Au, Av are linearly independent.

d) u, v are linearly dependent, but Au, v are linearly independent.