Math 218D Problem Session: Week 7

Answer Key

1. Linear Regression

Let us find the line $y = Cx + D$ which best fits the data points $(1,3)$, $(2,2)$, $(-2,1)$ (in the least-squares sense). If these points were collinear, then the coefficients C and D would solve the equation

$$A\begin{pmatrix} C \\ D \end{pmatrix} = b \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

But there is no solution to this system, as these 3 data points are not collinear. Instead, we will find the *least-squares solution* $\widehat{x} = \binom{C}{D}$, i.e., the solution of

$$A^T A\widehat{x} = A^T b.$$

a) Compute $A^T A$ and $A^T b$, and solve for the least-squares solutions $\widehat{x} = \binom{C}{D}$.

b) Plot the data points and the least-squares line $y = Cx + D$.

c) Where do the numbers in the vector $b - A\widehat{x}$ appear in the picture?

d) Compute the *error* $\|b - A\widehat{x}\|$.

Solution.

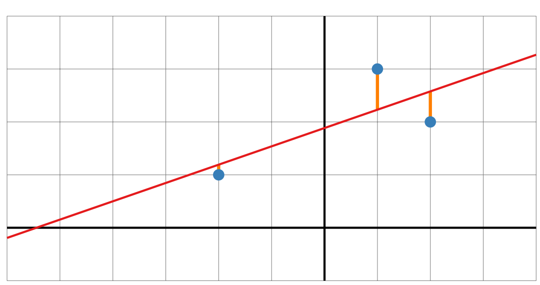
a)
$$A^T A = \begin{pmatrix} 9 & 1 \\ 1 & 3 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

We solve the system $A^T A\widehat{x} = A^T b$ by forming an augmented matrix and performing Gauss–Jordan elimination:

$$\left(\begin{array}{cc|c} 9 & 1 & 5 \\ 1 & 3 & 6 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 9/26 \\ 0 & 1 & 49/26 \end{array}\right).$$

The least-squares solution is $C = 9/26$, $D = 49/26$.

b) We plot the data points and the least-squares line $y = \frac{9}{26}x + \frac{49}{26}$. It may help to note that this line has x-intercept $-49/9 \approx -5.44$ and y-intercept $49/26 \approx 1.88$



c) $$b - A\hat{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 9/26 \\ 49/26 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 58/26 \\ 67/26 \\ 31/26 \end{pmatrix} = \frac{1}{26}\begin{pmatrix} 20 \\ -15 \\ -5 \end{pmatrix}.$$

The numbers in this vector are the vertical distances between the data points and the best-fit line, drawn in orange in the diagram.

d) $$\|b - A\hat{x}\| = \frac{1}{26}\sqrt{20^2 + (-15)^2 + (-5)^2} = \frac{5\sqrt{26}}{26} \approx 0.9806$$

2. Least Squares Practice

Find all least-squares solutions $\widehat{x}$ of each of the following systems of equations $Ax = b$, and compute the projection b_V of b onto $V = \mathrm{Col}(A)$ and the minimum value of $\|A\widehat{x} - b\|$.

a) $\begin{pmatrix} 0 & 2 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

c) $\begin{pmatrix} 8 & 2 & 3 \\ -3 & 4 & 4 \\ -2 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} -11 \\ 4 \\ 2 \end{pmatrix}$

Solution.

a) $\widehat{x} = \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix}$ $b_V = \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$ $\|A\widehat{x} - b\| = 2$

b) $\widehat{x} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$ $b_V = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\|A\widehat{x} - b\| = 0$

c) $\widehat{x} = \begin{pmatrix} 0 \\ 14 \\ -13 \end{pmatrix}$ $b_V = \begin{pmatrix} -11 \\ 4 \\ 2 \end{pmatrix}$ $\|A\widehat{x} - b\| = 0$

3. Orthogonal matrices

An *orthogonal matrix* is a *square* matrix Q such that $Q^T Q = I_n$. Which of the following matrices are orthogonal?

$$\textbf{a)} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solution.

a) $Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not an orthogonal matrix:

$$Q^T Q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) $Q = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ is not an orthogonal matrix:

$$Q^T Q = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

c) $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an orthogonal matrix:

$$Q^T Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

4. Rotation and reflection

A *rotation matrix* $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is an example of an orthogonal matrix.

a) Verify that R_θ is an orthogonal matrix by checking $R_\theta^T R_\theta = I_2$.

b) Draw the vectors $R_{\pi/6}\binom{1}{0}$ and $R_{\pi/6}\binom{1}{1}$.

c) Using dot products, compute the angle between the rotated vectors $R_{\pi/6}\binom{1}{0}$ and $R_{\pi/6}\binom{1}{1}$. Confirm that this is the same as the angle between the two vectors $\binom{1}{0}$ and $\binom{1}{1}$.

This is an example of a general phenomenon: multiplying by an orthogonal matrix preserves angles and lengths.

Consider a line $L = \text{Span}\{v\}$ in $\mathbf{R}^3$, and the orthogonal complement plane $V = L^\perp$. The *reflection matrix* for *reflection across* V is the orthogonal matrix

$$Q = I_3 - 2P_L,$$

where P_L is the projection matrix for L.

d) When $L = \text{Span}\{(0,0,1)\}$, compute the reflection matrix Q. Draw the line L and the plane V. Draw the vector $(1,-1,1)$, and compute and draw the projection $P_L(1,-1,1)$ and the reflection $Q(1,-1,1)$.

e) Confirm that *any* reflection matrix $Q = I_3 - 2P_L$ is an orthogonal matrix by showing that $Q^TQ = (I_3 - 2P_L)^T(I_3 - 2P_L)$ equals I_3.
[**Hint:** Remember that $P_L^2 = P_L$ and $P_L^T = P_L$.]

Solution.

a)
$$R_\theta^T R_\theta = \begin{pmatrix} \cos(\theta)^2 + \sin(\theta)^2 & 0 \\ 0 & \cos(\theta)^2 + \sin(\theta)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

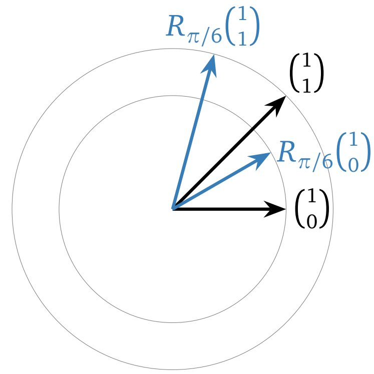
b) Using the identities $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and $\sin(\frac{\pi}{6}) = \frac{1}{2}$, we have

$$R_{\pi/6} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}.$$

Hence

$$R_{\pi/6}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \qquad R_{\pi/6}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (\sqrt{3}-1)/2 \\ (\sqrt{3}+1)/2 \end{pmatrix}.$$

These vectors are drawn below.



c) The angle between $(1,0)$ and $(1,1)$ is $\pi/4$. We want to confirm that the angle between $u = (\sqrt{3}/2, 1/2)$ and $v = ((\sqrt{3}-1)/2, (\sqrt{3}+1)/2)$ is $\pi/4$. We compute $u \cdot v = 1$, $\|u\| = 1$, and $\|v\| = \sqrt{2}$, so the angle in question is

$$\cos^{-1}\left(\frac{u \cdot v}{\|u\|\,\|v\|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4},$$

as it should be.

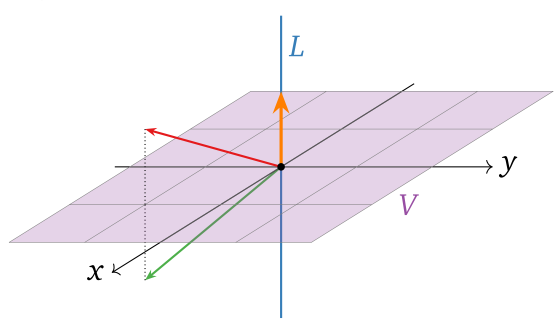
d) The projection matrix is

$$P_L = \frac{\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix}}{\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}} = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix},$$

so the reflection matrix is

$$Q = I_3 - 2P_L = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}.$$

The line L is the z-axis, so the plane V is the xy-plane. The vector $(1,-1,1)$ is drawn in red; the projection of $(1,-1,1)$ on to the z-axis is $P_L(1,-1,1) = (0,0,1)$ (in orange). The reflection of $(1,-1,1)$ across the xy-plane is $Q(1,-1,1) = (1,-1,-1)$ (in green).



e) We compute

$$
\begin{aligned}
Q^T Q &= (I_3 - 2P_L)^T (I_3 - 2P_L) = (I_3^T - 2P_L^T)(I_3 - 2P_L) \\
&= I_3 - 2P_L^T - 2P_L + 4P_L^T P_L
\end{aligned}
$$

Since $P_L^T = P_L$, this becomes $I_3 - 4P_L + 4(P_L)^2$. Since $P_L^2 = P_L$, this becomes $I_3 - 4P_L + 4P_L = I_3$. This shows that $Q^T Q = I_3$.