Math 218D Problem Session: Week 8

Answer Key

1. Gram-Schmidt and QR

The purpose of the Gram–Schmidt process is to replace a basis $\{v_1, \ldots, v_k\}$ of a subspace V of $\mathbf{R}^n$ with an **orthogonal basis** of V (a basis whose vectors are an orthogonal set).

The vectors $v_1 = (1, 2, -2)$, $v_2 = (1, 1, 1)$ form a basis for a plane V in $\mathbf{R}^3$. Set

$$u_1 = v_1$$
$$u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1.$$

These two vectors are the output of the Gram–Schmidt process.

a) Compute $\frac{u_1}{\|u_1\|}$ and $\frac{u_2}{\|u_2\|}$, and confirm that $\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}\right\}$ is an orthonormal set of vectors (you need to compute 3 dot products).

b) We can find the QR decomposition of $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}$ by setting

$$Q = \begin{pmatrix} | & | \\ \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \\ | & | \end{pmatrix}.$$

Then $A = QR$ for some upper-triangular matrix R, and you saw a formula for R in lecture. Here is another way to find R:

$$R = Q^T A.$$

Use this to compute R, and confirm that $A = QR$ by multiplying Q times R.

Note: The method of finding R given in lecture is much faster, as it involves only book-keeping your work from finding Q.

c) Explain why this formula for R worked, i.e. why $A = QR$ had to imply that $Q^T A = R$.

d) Explain how you could compute the projection matrix P_V using Q. (You do not need to do the computation.)

e) Find the least-squares solution of $Ax = (1, 1, 0)$ using $R\widehat{x} = Q^T b$.

Solution.

a) We calculate

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \qquad u_2 = \frac{1}{9}\begin{pmatrix} 8 \\ 7 \\ 11 \end{pmatrix}.$$

Noting that the unit vector in the direction of u_2 is the same as the unit vector in the direction of $(8,7,11)$, we have

$$\frac{u_1}{\|u_1\|} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \qquad \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{234}}\begin{pmatrix} 8 \\ 7 \\ 11 \end{pmatrix}.$$

These vectors have unit length, and $(1,2,-2)$ is orthogonal to $(8,7,11)$, so these two vectors are orthonormal.

b) We have

$$Q = \begin{pmatrix} 1/3 & 8/\sqrt{234} \\ 2/3 & 7/\sqrt{234} \\ -2/3 & 11/\sqrt{234} \end{pmatrix},$$

and we compute

$$R = Q^T A = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 8/\sqrt{234} & 7/\sqrt{234} & 11/\sqrt{234} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1/3 \\ 0 & 26/\sqrt{234} \end{pmatrix}.$$

c) If $A = QR$, then $Q^T A = Q^T QR$. Since $Q^T Q = I_2$, this simplifies to $Q^T A = R$.

d) $P_V = QQ^T$.

e) We compute

$$Q^T b = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 8/\sqrt{234} & 7/\sqrt{234} & 11/\sqrt{234} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 15/\sqrt{234} \end{pmatrix}.$$

Hence $R\widehat{x} = Q^T b$ becomes the system of equations

$$\begin{aligned} 3x_1 + \tfrac{1}{3}x_2 &= 1 \\ \tfrac{26}{\sqrt{234}}x_2 &= \tfrac{15}{\sqrt{234}}. \end{aligned}$$

Back-substitution gives $x_2 = 15/26$ and $x_1 = 7/26$:

$$\widehat{x} = \frac{1}{26}\begin{pmatrix} 7 \\ 15 \end{pmatrix}.$$

2. Another Gram–Schmidt

a) Apply the Gram–Schmidt process to the vectors

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

to obtain an orthogonal set $\{u_1, u_2, u_3\}$.

b) Find the QR decomposition of $A = \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$.

c) Consider the vector $b = (1,1,1)$. Since $\{u_1, u_2, u_3\}$ is a basis for $\mathbf{R}^3$, there are scalars x_1, x_2, x_3 such that $b = x_1u_1 + x_2u_2 + x_3u_3$. Solve for these scalars by taking the dot product of this equation with each of u_1, u_2, u_3, giving 3 equations

$$b \cdot u_i = (x_1u_1 + x_2u_2 + x_3u_3) \cdot u_i \quad \text{for} \quad i = 1,2,3.$$

(These equations simplify dramatically when you compute the dot products.)

d) Explain how you could instead solve for these scalars using the formula $QQ^T = P_{\mathbf{R}^3} = I_3$.
Hint: Note that $b = Q(Q^T b)$.

Solution.

a) We do Gram–Schmidt:

$$u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$u_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{(1,1,0)\cdot(1,0,1)}{(1,1,0)\cdot(1,1,0)}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}$$

$$u_3 = \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{(1,1,0)\cdot(0,1,1)}{(1,1,0)\cdot(1,1,0)}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{(1/2,-1/2,1)\cdot(0,1,1)}{(1/2,-1/2,1)\cdot(1/2,-1/2,1)}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix} = \begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix}.$$

b) The orthonormal vectors are $\frac{1}{\sqrt{2}}(1,1,0)$, $\frac{1}{\sqrt{6}}(1,-1,2)$, $\frac{1}{\sqrt{3}}(-1,1,1)$. Therefore

$$Q = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{pmatrix}.$$

We compute R using the formula:

$$R = \begin{pmatrix} \|u_1\| & \frac{v_2\cdot u_1}{u_1\cdot u_1}\|u_1\| & \frac{v_3\cdot u_1}{u_1\cdot u_1}\|u_1\| \\ 0 & \|u_2\| & \frac{v_3\cdot u_2}{u_2\cdot u_2}\|u_2\| \\ 0 & 0 & \|u_3\| \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix}$$

c) These equations simplify to

$$\begin{aligned} b\cdot u_1 &= \|u_1\|^2 x_1, \\ b\cdot u_2 &= \|u_2\|^2 x_2, \\ b\cdot u_3 &= \|u_3\|^2 x_3, \end{aligned}$$

since $\{u_1, u_2, u_3\}$ is orthogonal. Now it is easy to solve for x_1, x_2, x_3:

$$\begin{aligned} x_1 &= \frac{b\cdot u_1}{\|u_1\|^2} = \frac{2}{2} = 1, \\ x_2 &= \frac{b\cdot u_2}{\|u_2\|^2} = \frac{1}{3/2} = \frac{2}{3}, \\ x_3 &= \frac{b\cdot u_3}{\|u_3\|^2} = \frac{2/3}{4/3} = \frac{1}{2}. \end{aligned}$$

In other words,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

d) First we compute

$$Q^T b = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{2} \\ 2/\sqrt{6} \\ 1/\sqrt{3} \end{pmatrix}.$$

Since $QQ^T = P_{\mathbf{R}^3} = I_3$ we have $b = Q(Q^T b)$. On the other hand, the columns of Q are the vectors $\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}$, so

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = Q(Q^T b) = Q\begin{pmatrix} 2/\sqrt{2} \\ 2/\sqrt{6} \\ 1/\sqrt{3} \end{pmatrix} = \frac{2}{\sqrt{2}}\frac{u_1}{\|u_1\|} + \frac{2}{\sqrt{6}}\frac{u_2}{\|u_2\|} + \frac{1}{\sqrt{3}}\frac{u_3}{\|u_3\|}.$$

Substituting $\|u_1\| = \sqrt{2}$, $\|u_2\| = \sqrt{6}/2$, $\|u_3\| = 2/\sqrt{3}$ gives

$$b = u_1 + \frac{2}{3}u_2 + \frac{1}{2}u_3,$$

as before.

3. Some quick determinants

Compute the determinants of the following matrices:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 10 & 17 \\ 0 & 2 & \pi \\ 0 & 0 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$ **e)** $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **f)** $\begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 5 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ **h)** $\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}^{20}$

Solution.

a) 0 **b)** 6 **c)** 5

d) -5 **e)** 1 **f)** 24

g) 18 **h)** $(-1)^{20} = 1$

4. Some determinants with variables

a) Compute the determinant of each of $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$, A^2, A^{-1}, and $A - xI_2$. Find the two values of x so that $\det(A - xI_2) = 0$.

b) Compute the determinant of

$$\begin{pmatrix} 1-x & 1 & 1 \\ 2 & 2-x & 2 \\ 1 & 2 & 3-x \end{pmatrix}.$$

This is a polynomial in the variable x—what degree is the polynomial?

Solution.

a) We have $\det(A) = -2$, so $\det(A^2) = \det(A)^2 = 4$ and $\det(A)^{-1} = 1/\det(A) = -\frac{1}{2}$. Using the formula for a 2×2 determinant, we get

$$\det(A - xI_2) = \det\begin{pmatrix} 1-x & 2 \\ 3 & 4-x \end{pmatrix} = x^2 - 5x - 2.$$

Using the quadratic formula, we find the solutions of $x^2 - 5x - 2 = 0$ to be $x = \frac{1}{2}(5 \pm \sqrt{33})$.

b) We perform Gaussian elimination, being careful always to choose a pivot with known entries:

$$\begin{pmatrix} 1-x & 1 & 1 \\ 2 & 2-x & 2 \\ 1 & 2 & 3-x \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3-x \\ 2 & 2-x & 2 \\ 1-x & 1 & 1 \end{pmatrix}$$
$$\xrightarrow{R_2 -= 2R_1} \begin{pmatrix} 1 & 2 & 3-x \\ 0 & -x-2 & -4+2x \\ 1-x & 1 & 1 \end{pmatrix}$$
$$\xrightarrow{R_3 -= (1-x)R_1} \begin{pmatrix} 1 & 2 & 3-x \\ 0 & -x-2 & -4+2x \\ 0 & 2x-1 & -x^2+4x-2 \end{pmatrix}$$
$$\xrightarrow{R_3 += 2R_2} \begin{pmatrix} 1 & 2 & 3-x \\ 0 & -x-2 & -4+2x \\ 0 & -5 & -x^2+8x-10 \end{pmatrix}$$
$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3-x \\ 0 & -5 & -x^2+8x-10 \\ 0 & -x-2 & -4+2x \end{pmatrix}$$
$$\xrightarrow{R_3 -= \frac{x+2}{5}R_2} \begin{pmatrix} 1 & 2 & 3-x \\ 0 & -5 & -x^2+8x-10 \\ 0 & 0 & -4+2x-\frac{(x+2)}{5}(-x^2+8x-10) \end{pmatrix}.$$

Since we performed row replacements and two row swaps, we have

$$\det\begin{pmatrix} 1-x & 1 & 1 \\ 2 & 2-x & 2 \\ 1 & 2 & 3-x \end{pmatrix}$$
$$= (-1)^2 \det\begin{pmatrix} 1 & 2 & 3-x \\ 0 & -5 & -x^2+8x-10 \\ 0 & 0 & -4+2x-\frac{(x+2)}{5}(-x^2+8x-10) \end{pmatrix}$$
$$= -5(-4+2x)+(x+2)(-x^2+8x-10) = -x^3+6x^2-x.$$

This is a polynomial of degree 3.

5. Signs of determinants

We gave a geometric interpretation of the absolute value of a determinant in lecture. In this problem we will investigate what the *sign* of a determinant means geometrically. (The *sign* of a number is $+1$ if the number is positive and -1 if it is negative.)

a) Draw the vectors $u = (1, -1)$, $v = (2, 3)$. Is v clockwise or counterclockwise from u? What is the *sign* of the determinant of $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$?

b) Draw the vectors $u = (-1, 2)$, $v = (1, 1)$. Is v clockwise or counterclockwise from u? What is the *sign* of the determinant of $\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$?

Let u, v, w be vectors in $\mathbf{R}^3$. With your right hand, point your index finger in the direction of u, your middle finger in the direction of v, and your thumb in the direction of w. We say that u, v, w are in *right-hand order* if, when you point your thumb at your face, your middle finger is counterclockwise of your index finger. Otherwise, the vectors are in *left-hand order*.

c) Are the vectors $u = (0, 1, 0)$, $v = (1, 1, 0)$, $w = (1, 1, 1)$ in right-hand order or left-hand order?

d) Are the vectors $u = (1, 1, 0)$, $v = (0, 1, 0)$, $w = (1, 1, 1)$ in right-hand order or left-hand order?

e) What is the sign of the determinants of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}?$$

f) What do you think the sign of the 3×3 determinant has to do with right-hand order?

Solution.

a) The vector v is counterclockwise from u. The sign of the determinant is $+1$.

b) The vector v is clockwise from u. The sign of the determinant is -1.

c) The vectors $u = (0, 1, 0)$, $v = (1, 1, 0)$, $w = (1, 1, 1)$ are in left-hand order.

d) The vectors $u = (1, 1, 0)$, $v = (0, 1, 0)$, $w = (1, 1, 1)$ are in right-hand order.

e) The sign of the determinants of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

are -1 and $+1$.

f) The sign of a 3×3 determinant is $+1$ if the rows are in right-hand order, and -1 if the rows are in left-hand order.