Math 218D Problem Session: Week 9

October 28, 2022

1. Some simple examples

For each of the following matrices A,

i) Find the characteristic polynomial $p(\lambda) = \det(A - \lambda I_2)$.

ii) Find all the *eigenvalues* by solving $p(\lambda) = 0$.

iii) For each eigenvalue λ_i, find a basis of the associated *eigenspace* $\mathrm{Nul}(A - \lambda_i I_2)$.

iv) An $n \times n$ matrix A is diagonalizable if and only if the dimensions of the eigenspaces add up to n. For these matrices, you may have one or two eigenspaces, depending on how many different roots $p(\lambda)$ has. Is the matrix A diagonalizable? Is the matrix A diagonal?

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ **c)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **d)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ **f)** $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ **g)** $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

2. A 2×2 diagonalization

Consider the matrix $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$.

a) Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I_2)$.

b) Using the quadratic formula, find the two solutions to $p(\lambda) = 0$. The two solutions, λ_1 and λ_2, are the two eigenvalues of A.

c) Find the eigenvector $v_1 = (x_1, y_1)$ by solving the eigenvector equation

$$(A - \lambda_1 I_2)v_1 = 0$$

Note that there is more than one solution—choose any non-zero solution.

d) Find the eigenvector $v_2 = (x_2, y_2)$ by solving the eigenvector equation

$$(A - \lambda_2 I_2)v_2 = 0.$$

e) Diagonalize A, by making a matrix of eigenvalues $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, a matrix of eigenvectors $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, and confirming that $A = CDC^{-1}$ by multiplying these three matrices.

f) Compute the vector $A^n(1,2)$.

Hint: Find scalars c_1, c_2 so that $(1,2) = c_1 v_1 + c_2 v_2$. It may help to use the matrix C^{-1} to do this. Then use the formula $A^n(c_1 v_1 + c_2 v_2) = c_1 A^n v_1 + c_2 A^n v_2$.

g) When n is very large, $\|A^{n+1}(1,2)\| / \|A^n(1,2)\|$ is approximately ______.

h) When n is very large, $\|A^{n+1}(1,1)\| / \|A^n(1,1)\|$ is approximately ______ (this should be easier than **g)**.)

i) If you were given a random vector w, what would you expect $\|A^{n+1}w\| / \|A^n w\|$ to approximate when n is very large?

3. Some 3×3 characteristic polynomials

Compute the characteristic polynomials and eigenvalues of the matrices

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Decide if each matrix is diagonalizable, and if it is, diagonalize it.

4. Traces and determinants

Recall that the trace $\text{Tr}(A)$ is the sum of the diagonal entries of A.

a) For each of the matrices in problem 1(a)–(f), factor $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Verify that

$$\text{Tr}(A) = \lambda_1 + \lambda_2 \text{ and } \det(A) = \lambda_1 \cdot \lambda_2.$$

b) For any $n \times n$ matrix, the polynomial $p(\lambda) = \det(A - \lambda I_n)$ can be factored as

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)\cdots(\lambda - \lambda_n).$$

Verify that

$$\det(A) = \lambda_1 \cdots \lambda_n.$$

Hint: What happens to $\det(A - \lambda I_n)$ when you set $\lambda = 0$? What happens to $(-1)^n(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$ when you set $\lambda = 0$?

c) The determinant $\det(A)$ has another product formula:

$$\det(A) = (-1)^k d_1 \cdots d_n,$$

when the A has REF with pivot entries $d_1, \ldots, d_n$, found using Gaussian elimination w/o row scaling and with k row swaps. Even though this formula looks quite similar to the formula of **b)**, eigenvalues and pivots are not at all the same.

Find an example of a 2×2 matrix where the pivots d_1, d_2 are not the same as the eigenvalues λ_1, λ_2.

d) **(Challenge)** For any $n \times n$ matrix, show that $\text{Tr}(A) = \lambda_1 + \cdots + \lambda_n$.

5. Linear independence of eigenvectors

a) Consider a matrix A with two distinct eigenvalues $\lambda_1 \neq \lambda_2$, with associated eigenvectors v_1 and v_2. Show that v_1 is not a scalar multiple of v_2.

Hint: Suppose they were scalar multiples, $v_1 = cv_2$. What happens when you multiply this equation by A?

b) Consider a matrix A with three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$, with associated eigenvectors v_1, v_2 and v_3. Show that v_1, v_2, and v_3 are linearly independent.

Hint: Suppose they were dependent, $a_1v_1 + a_2v_2 + a_3v_3 = 0$, with $a_3 \neq 0$. Multiply this equation by A. Can you get a new linear dependence where $a_3 = 0$?