

MATH 218D-1
PRACTICE FINAL EXAMINATION

Name		Duke NetID	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. You may bring a 8.5×11 -**inch note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

Problem 1.

[25 points]

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{pmatrix}.$$

a) Find the singular values of A .

$$\sigma_1 = \boxed{\sqrt{10}} \quad \sigma_2 = \boxed{\sqrt{5}}$$

b) Find the singular value decomposition of A in outer product form.

$$A = \sqrt{10} \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 6/5\sqrt{2} & 3/5\sqrt{2} & -1/5\sqrt{2} & 2/5\sqrt{2} \end{pmatrix} \\ + \sqrt{5} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/5 & 4/5 & 2/5 & 1/5 \end{pmatrix}$$

Now we change matrices to avoid carry-through error. Consider the 2×4 matrix with singular value decomposition

$$A = 3\sqrt{10} \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} + 2\sqrt{10} \begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

c) Find the singular value decomposition of A in matrix form: $A = U\Sigma V^T$ for

$$U = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 3\sqrt{10} & 0 & 0 & 0 \\ 0 & 2\sqrt{10} & 0 & 0 \end{pmatrix} \\ V = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \end{pmatrix}$$

d) What is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$, and at which vectors is the maximum attained?

$$\text{value} = \boxed{3\sqrt{10}} \quad \text{vectors} = \pm \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

e) Write the SVD of the pseudo-inverse $\boxed{\text{of } A^T}$ in outer product form.

$$(A^T)^+ = \frac{1}{3\sqrt{10}} \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2\sqrt{10}} \begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

f) Compute the shortest least-squares solution of $A^T x = b$ for $b = (0, 8, 4, 0)$.

[Hint: do not expand out your answer to \mathbf{e} .]

$$\hat{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

g) Compute the matrix for projection onto $V = \text{Row}(A)$.

$$P_V = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Problem 2.

[15 points]

An online movie-streaming service collects star ratings from its viewers and uses these to predict what movies you will like based on your previous ratings. The following are the ratings that ten (fictitious) people gave to three (fictitious) movies, on a scale of 0–10:

	Abe	Amy	Ann	Ben	Bob	Eve	Dan	Don	Ian	Meg
<i>Prognosis Negative</i>	7.8	6.1	2.4	9.8	10	3.0	6.3	3.6	6.7	6.3
<i>Ponce De Leon</i>	6.0	7.9	6.4	8.1	7.1	6.4	7.3	7.9	6.2	8.1
<i>Lenore's Promise</i>	5.8	8.2	8.2	6.8	6.2	8.7	7.3	9.2	6.8	8.2

We put these data into a matrix:

$$A_0 = \begin{pmatrix} 7.8 & 6.1 & 2.4 & 9.8 & 10 & 3.0 & 6.3 & 3.6 & 6.7 & 6.3 \\ 6.0 & 7.9 & 6.4 & 8.1 & 7.1 & 6.4 & 7.3 & 7.9 & 6.2 & 8.1 \\ 5.8 & 8.2 & 8.2 & 6.8 & 6.2 & 8.7 & 7.3 & 9.2 & 6.8 & 8.2 \end{pmatrix}.$$

The row averages are 6.2, 7.14, and 7.54, respectively; subtracting them gives the recentered data matrix A . The normalized recentered data matrix has singular value decomposition

$$\frac{1}{\sqrt{10-1}}A \approx 2.78 \begin{pmatrix} 0.938 \\ 0.046 \\ -0.343 \end{pmatrix} v_1^T + 1.02 \begin{pmatrix} 0.168 \\ 0.805 \\ 0.569 \end{pmatrix} v_2^T + 0.182 \begin{pmatrix} 0.302 \\ -0.591 \\ 0.747 \end{pmatrix} v_3^T.$$

- a) Find a basis for the line of best fit for our recentered data points. What is the variance along this line?

$$\text{basis} = \left\{ \begin{pmatrix} 0.938 \\ 0.046 \\ -0.343 \end{pmatrix} \right\} \quad \text{variance} = \boxed{7.73}$$

- b) Find an orthonormal basis for the plane of best fit V .

$$\left\{ \begin{pmatrix} 0.938 \\ 0.046 \\ -0.343 \end{pmatrix}, \begin{pmatrix} 0.168 \\ 0.805 \\ 0.569 \end{pmatrix} \right\}$$

- c) What is the variance along V ? What is the variance along V^\perp ?

$$\text{variance along } V = \boxed{8.77} \quad \text{variance along } V^\perp = \boxed{0.0331}$$

- d) Using the fact that the left singular vectors are orthonormal, find an implicit equation for V of the form $x_3 = a_1x_1 + a_2x_2$.

$$V = \left\{ (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = \boxed{-0.404}x_1 + \boxed{0.791}x_2 \right\}$$

- e) Suppose that Joe gave *Prognosis Negative* a rating of 8.5 and *Ponce De Leon* a rating of 6.2. How would you expect Joe to rate *Lenore's Promise*?

$$\boxed{5.87} \text{ stars}$$

Problem 3.

[20 points]

Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$.

a) Find the QR decomposition of A .

$$Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{pmatrix} \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2}/2 \\ 0 & \sqrt{6}/2 \end{pmatrix}$$

b) Compute the SVD of R in outer product form.

$$R = \sqrt{3} \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} (1/\sqrt{2} \quad 1/\sqrt{2}) + 1 \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} (-1/\sqrt{2} \quad 1/\sqrt{2})$$

c) Explain why R and A have the same singular values and right-singular vectors.

$$\text{Because } A^T A = (QR)^T (QR) = R^T Q^T QR = R^T R.$$

d) Compute the SVD of A in outer-product form.

$$A = \sqrt{3} \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} (1/\sqrt{2} \quad 1/\sqrt{2}) + 1 \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} (-1/\sqrt{2} \quad 1/\sqrt{2})$$

e) Compute the orthogonal decomposition of $b = (2, 1, 2)$ with respect to $V = \text{Col}(A)$.

$$b_V = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad b_{V^\perp} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

f) Find a basis for $\text{Nul}(A^T)$.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Problem 4.

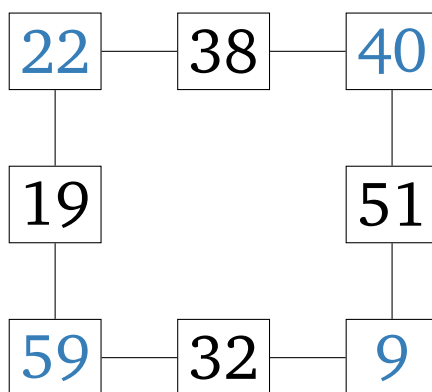
[20 points]

a) Find the LU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

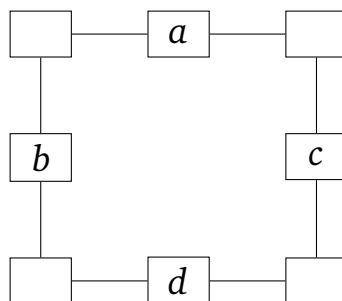
b) Fill in the blanks with positive whole numbers so that the sums of the numbers on the top, bottom, left, and right sides of the square all equal to 100.



c) Compute a basis for $\text{Nul}(A^T)$.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

d) What equation in a, b, c, d must be satisfied so that it is possible to fill in the blanks as in b)?



$$\boxed{-1} a + \boxed{1} b + \boxed{1} c + \boxed{-1} d = 0$$

Problem 5.

[20 points]

Consider the matrix

$$A = \begin{pmatrix} -3 & -8 & 14 \\ 0 & -7 & 12 \\ -1 & -7 & 12 \end{pmatrix}.$$

Its eigenvalues are 2, 1, and -1 .

- a) Compute an eigenbasis for A . Choose eigenvectors with integer (whole-number) entries.

$$2\text{-eigenvector} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \quad 1\text{-eigenvector} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad (-1)\text{-eigenvector} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

- b) Solve the difference equation

$$v_{k+1} = Av_k \quad v_0 = \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix}.$$

$$v_k = 2^k \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - (-1)^k \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

- c) As $k \rightarrow \infty$, the ratio of the entries of v_k tend to the following ratio:

$$\left(\boxed{2} : \boxed{4} : \boxed{3} \right)$$

- d) Solve the initial value problem

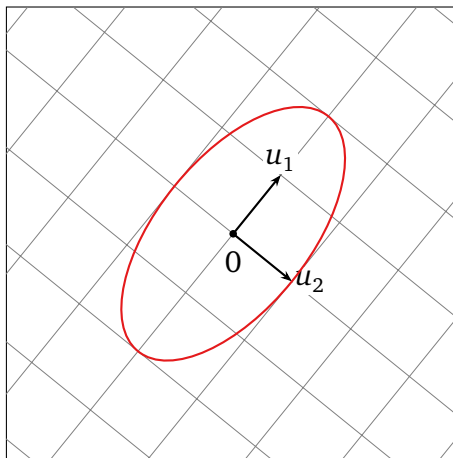
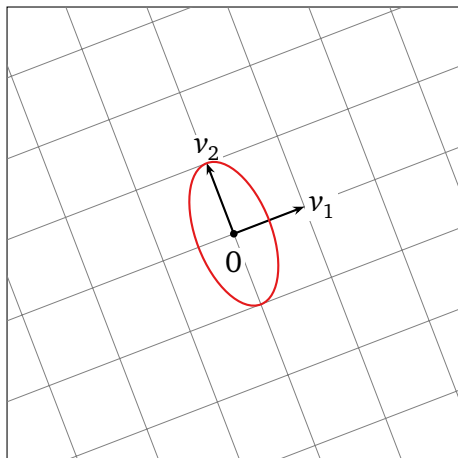
$$u' = Au \quad u(0) = \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix}$$

$$u(t) = e^{2t} \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - e^{-t} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Problem 6.

[16 points]

A certain 2×2 matrix A has singular values 2 and 1. Its right-singular vectors v_1, v_2 are drawn in the picture on the left, and its left-singular vectors u_1, u_2 are drawn in the picture on the right.



- a) Draw the set $\{Ax : \|x\| = 1\}$ in the picture on the right.
- b) Explain why $\|Ax\| = 1$ if and only if $q(x) = x^T S x = 1$, where S is the symmetric matrix $A^T A$.

Because

$$\|Ax\|^2 = (Ax) \cdot (Ax) = (Ax)^T (Ax) = x^T (A^T A)x.$$

- c) Draw the set $\{x : \|Ax\| = 1\} = \{x : q(x) = 1\}$ in the picture on the left.

Problem 7.

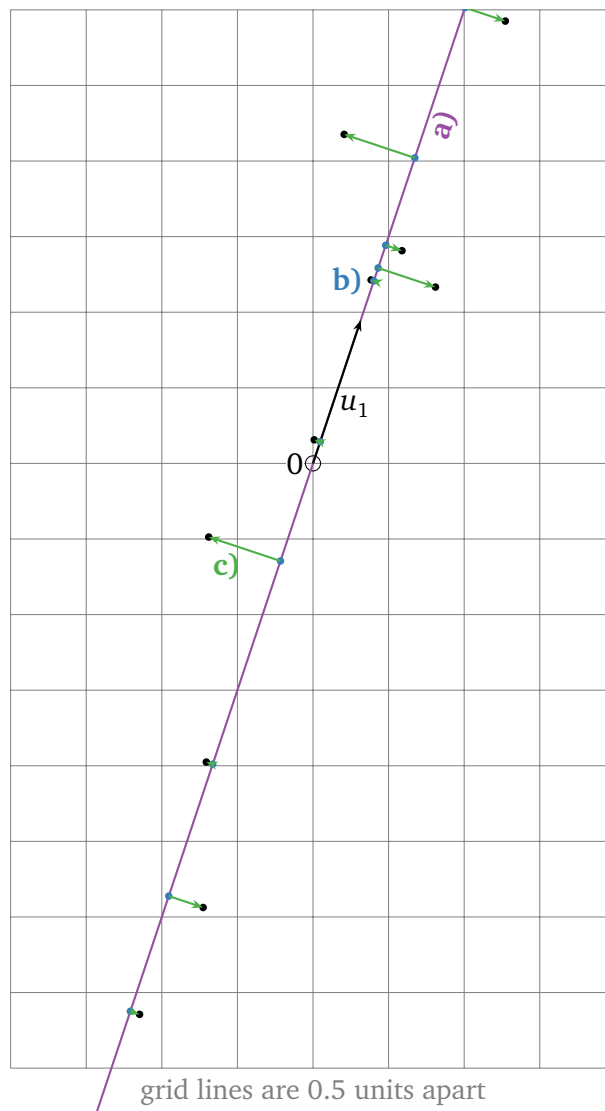
[20 points]

A centered data matrix A has 2 rows and 10 columns. Its SVD has the form

$$A = 7u_1v_1^T + 0.9u_2v_2^T,$$

where u_1, u_2 and v_1, v_2 are the singular vectors. The columns of A and the first left singular vector u_1 are drawn below. *Draw and label:*

- the best-fit line (for the columns of A) in the sense of orthogonal least squares;
- the columns of $7u_1v_1^T$ (drawn as dots);
- the columns of $.9u_2v_2^T$ (drawn as arrows).
- The square of the second singular value (0.9^2) is equal to what geometric quantity in your drawing?



Problem 8.

[20 points]

True/false problems: **circle** the correct answer. No justification is needed.

All matrices in this problem have real entries.

- a) T F If $Ax = b$ has at least one solution for every $b \in \mathbf{R}^m$, then A has full row rank.
- b) T F If x is a solution of $Ax = b$, then every vector in $\text{Span}\{x\}$ is also a solution of $Ax = b$.
- c) T F The maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$ is the largest eigenvalue of A .
- d) T F A triangular matrix with real entries can have a complex (non-real) eigenvalue.
- e) T F If S is a symmetric $n \times n$ matrix and $x, y \in \mathbf{R}^n$, then $(Sx) \cdot y = x \cdot (Sy)$.
- f) T F Every unit vector is a right-singular vector of an orthogonal matrix.
- g) T F The singular values of a diagonalizable, invertible, square matrix are the absolute values of its eigenvalues.
- h) T F If A is a diagonalizable matrix with characteristic polynomial $(1 - \lambda)^n$, then $A = I_n$.
- i) T F If λ^2 is an eigenvalue of A^2 , then λ is an eigenvalue of A .
- j) T F If A is invertible, then $\text{Col}(BA) = \text{Col}(B)$.

Problem 9.

[24 points]

Short-answer problems: no justification is necessary.

a) Suppose that

$$A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \quad A \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \quad A \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix}.$$

Compute the characteristic polynomial of A .

$$p(\lambda) = (-1 - \lambda)(2 - \lambda)(-2 - \lambda)$$

b) If A is a 20×60 matrix, then $\text{rank}(A) + \dim \text{Nul}(A) = \boxed{60}$.

c) Let A be an $n \times n$ matrix. Which of the following are *equivalent* to the statement “ A is diagonalizable over the real numbers?” (Fill in the circles of all that apply.)

- $A = CDC^{-1}$ for an invertible real matrix C and a diagonal real matrix D .
- Each eigenvalue of A has at least one eigenvector.
- For each real eigenvalue λ of A , the algebraic multiplicity of λ is equal to $\dim \text{Nul}(A - \lambda I_n)$.
- The (real) eigenvectors of A span \mathbf{R}^n .
- Zero is not an eigenvalue of A .

d) Let A be a 3×2 matrix with singular values 2 and 1.

$$\text{rank}(A) = \boxed{2} \quad \det(A^T A) = \boxed{4} \quad \det(AA^T) = \boxed{0}$$

e) Which of the following sets form a basis for the 9-eigenspace of $\begin{pmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{pmatrix}$?

(Fill in the circles of all that apply.)

- $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\}$
- $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$
- $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right\}$
- $\left\{ \begin{pmatrix} -7 \\ -2 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix} \right\}$
- $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} \right\}$
- $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

f) Which of the following are subspaces of \mathbf{R}^4 ? (Fill in the circles of all that apply.)

- The null space of a 3×4 matrix.
- The row space of a 3×4 matrix.
- The column space of a 3×4 matrix.
- An eigenspace of a 4×4 matrix.
- The span of 4 vectors in \mathbf{R}^5 .
- The orthogonal complement of a plane in \mathbf{R}^4 .

Problem 10.

[20 points]

Give an example of a matrix with each of the following properties, or explain why no such example exists. (No justification is needed if an example does exist.)

All matrices in this problem have real entries.

a) A symmetric matrix S satisfying

$$S \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad S \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0.$$

No such matrix exists: the 1-eigenspace must be orthogonal to the 0-eigenspace.

b) A 3×3 matrix A such that $\text{Nul}(A) = \text{Col}(A)$.

No such matrix exists: $\dim \text{Nul}(A) + \dim \text{Col}(A) = 3$.

c) A 3×4 matrix with orthogonal columns.

No such matrix exists: \mathbf{R}^3 does not contain four orthogonal (hence linearly independent) vectors.

d) A 2×2 symmetric matrix that is positive semidefinite but not positive definite.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

e) A 3×2 matrix with singular values 3, 2, and 1.

No such matrix exists: a 3×2 matrix cannot have rank 3.