MATH 218D-1
PRACTICE MIDTERM EXAMINATION 1

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -3 & -2 \\ 9 & 7 & 2 \end{pmatrix}.$$

a) Find a lower-unitriangular matrix L and a matrix U in REF such that $A = LU$.

$$L = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix} \qquad U = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

b) Use the LU decomposition you found in **a)** to solve the equation $Ax = (0, 2, 0)$.

$$x = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

c) Briefly explain why one would want to use an LU decomposition to solve $Ax = b$.

d) Express L as a product of three elementary matrices.

$$L = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}\begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}\begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

[Scratch work for Problem 1]

Problem 2.

[25 points]

Consider the system of equations

$$\begin{aligned} x_1 - 2x_2 \phantom{{}+x_3} - x_4 &= 2 \\ -x_1 + 2x_2 + x_3 + 2x_4 &= -3 \\ 3x_1 - 6x_2 - x_3 - 4x_4 &= 7. \end{aligned}$$

a) Express this system of equations as a matrix equation $Ax = b$:

$$\begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix} x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

b) Perform Gauss–Jordan elimination on the augmented matrix $\left(A \mid b\right)$ to produce a matrix in reduced row echelon form. Please write all row operations that you perform.

$$\left(\begin{array}{cccc|c} & & & & \\ & & & & \\ & & & & \end{array}\right)$$

c) The free variables(s) are $\boxed{}$ and the rank of A is $\boxed{}$.

[Scratch work for Problem 2]

(Problem 2, continued)

d) Write the solution set of this system of equations in parametric vector form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} +$$

e) The solution set is a (circle one) $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\boxed{}}$.

f) Find a basis for Nul(A).

$$\left\{ \qquad \right\}$$

g) Write down any *nontrival* solution of $Ax = 0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 2]

Problem 3.

[20 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad v_4 = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}.$$

a) Find a linear relation among v_1, v_2, v_3, v_4.

$$\boxed{0 = \qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

b) The set $\{v_1, v_2, v_3, v_4\}$ is (circle one) $\begin{pmatrix} \text{linearly dependent} \\ \text{linearly independent} \end{pmatrix}$.

c) Which one of the following two vectors is in $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$?
(Circle the correct one.)

$$u = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \qquad\qquad w = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

d) Express $(2, -3, 7)$ as a linear combination of v_1, v_2, v_3, v_4 in two different ways.

$$\begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix} = \boxed{} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \boxed{} \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} + \boxed{} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \boxed{} \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix} = \boxed{} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \boxed{} \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} + \boxed{} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \boxed{} \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$$

e) Which of the following sets form a basis for $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$?
(Circle all that apply.)

$$\{v_1, v_2\} \qquad \{v_1, v_2, v_3\} \qquad \{v_2, v_3\}$$

$$\{v_2, v_3, v_4\} \qquad \{v_3, v_4\} \qquad \{v_2, v_4\}$$

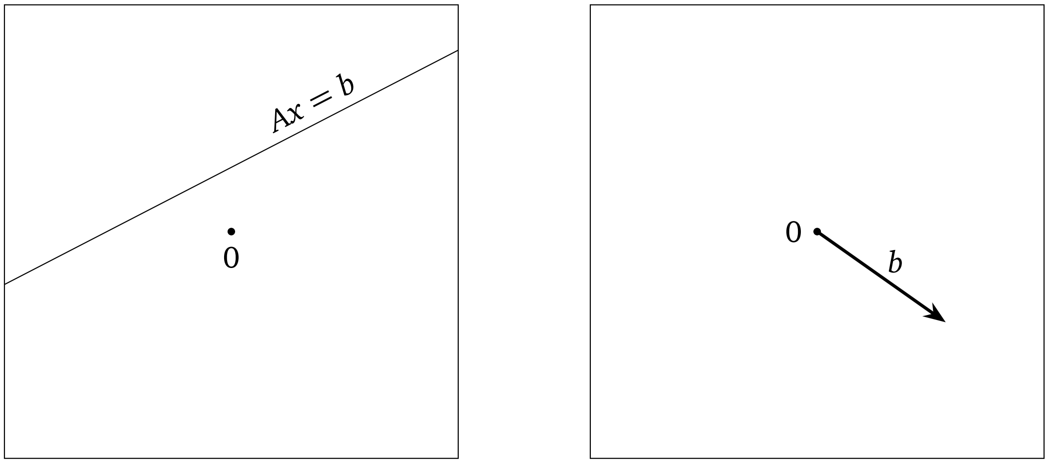
f) $\dim\big(\mathrm{Span}\{v_1, v_2, v_3, v_4\}\big) = \boxed{}$

[Scratch work for Problem 3]

Problem 4.

[15 points]

For a certain 2×2 matrix A and vector $b \in \mathbf{R}^2$, the solution set of $Ax = b$ is drawn in the picture on the left, and b is drawn in the picture on the right.



a) Draw Nul(A) in the picture on the left.

b) Draw Col(A) in the picture on the right.

c) Draw a vector b' in the picture on the right that makes the system $Ax = b'$ inconsistent.

[Scratch work for Problem 4]

Problem 5. [10 points]

For each of the following subsets,

- determine if the subset is a subspace.

If it is not a subspace,

- explain why not.

If it is a subspace,

- express it as the null space or the column space of a matrix, and
- find a basis.

a)
$$V_1 = \left\{ \begin{pmatrix} 2(a+c)-b \\ a-2b+c \end{pmatrix} \;:\; a,b,c \in \mathbf{R} \right\}$$

b)
$$V_2 = \left\{ \begin{pmatrix} a+b \\ 2 \\ 2a-2b \end{pmatrix} \;:\; a,b,c \in \mathbf{R} \right\}$$

c)
$$V_3 = \text{the } x\text{-axis in the } xy\text{-plane}$$

d)
$$V_4 = \text{all vectors } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ such that } x \geq y$$

[Scratch work for Problem 5]

Problem 6.

[15 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

a) A 2×2 matrix whose column space is a point.

b) A 2×2 matrix whose null space is *all of* $\mathbf{R}^2$.

c) An invertible 3×3 matrix A such that the solution set of $Ax = 0$ is the x-axis.

d) A 2×2 matrix whose columns are linearly *dependent*.

e) A 2×2 matrix A such that the solution set of $Ax = \binom{1}{2}$ is the point $\left\{\binom{-1}{1}\right\}$, and such that the system $Ax = \binom{2}{1}$ is inconsistent.

[Scratch work for Problem 6]