MATH 218D-1
MIDTERM EXAMINATION 1

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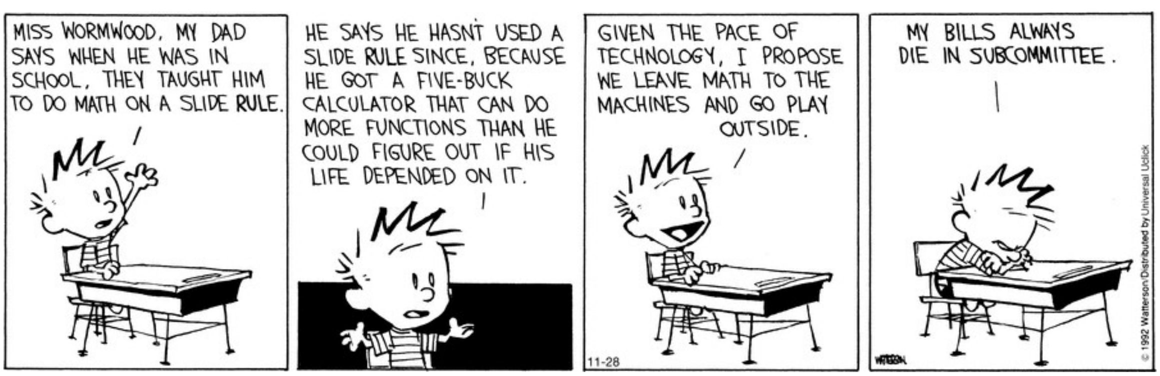
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 6 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

a) Find a $PA = LU$ decomposition of A.

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

(There are other correct answers.)

b) Solve $Ax = b$ for $b = (1, 8, 4)$ using your answer to **a)**.

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

c) Compute A^{-1}. Please write the row operations you performed.

$$A^{-1} = \begin{pmatrix} -3 & -1/2 & 2 \\ 1 & 1/2 & -1 \\ 0 & -1/2 & 1 \end{pmatrix}$$

d) Solve $Ax = b$ for an unknown vector $b = (b_1, b_2, b_3)$. Your answer will be a formula in terms of b_1, b_2, b_3.

$$x = \begin{pmatrix} -3b_1 - \frac{1}{2}b_2 + 2b_3 \\ b_1 + \frac{1}{2}b_2 - b_3 \\ -\frac{1}{2}b_2 + b_3 \end{pmatrix}$$

e) Express A^{-1} as a product of elementary matrices. (Write *matrices*, not row operations.)

$$A^{-1} = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

f) Express A as a product of elementary matrices.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2. [15 points]

Consider the matrix equation $Ax = b$ for

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 3 & 9 & -2 & 8 \\ 2 & 6 & 2 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

a) Find the parametric vector form of the solution set of $Ax = b$.

$$x = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

b) Compute the following quantities:

$$\operatorname{rank}(A) = \boxed{2} \qquad \dim \operatorname{Nul}(A) = \boxed{2} \qquad \dim \operatorname{Col}(A) = \boxed{2}$$

$$\dim \operatorname{Row}(A) = \boxed{2} \qquad \dim \operatorname{Nul}(A^T) = \boxed{1}.$$

c) Find a basis for Nul(A).

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

d) Given that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix},$$

express the solution set of $Ax = (2, 2, 8)$ as a translate of a span.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} + \operatorname{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

e) Find a basis for Col(A).

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \right\}$$

Problem 3. [15 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 1 \\ -7 \\ -7 \end{pmatrix}.$$

a) Find a linear relation among v_1, v_2, v_3.

$$\boxed{5}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \boxed{1}\begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + \boxed{1}\begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} = 0$$

b) Span$\{v_1, v_2, v_3\}$ is a (circle one) (line / plane / space) in (fill in the blank) $\mathbf{R}^{\boxed{3}}$.

Answer: plane

c) Is $v_4 \in \text{Span}\{v_1, v_2, v_3\}$? If so, express v_4 as a linear combination of v_1, v_2, v_3.

(Yes / No) Answer: No

$$v_4 = \boxed{}\,v_1 + \boxed{}\,v_2 + \boxed{}\,v_3$$

d) Is $\{v_1, v_2, v_3, v_4\}$ linearly dependent? If so, find a linear relation among v_1, v_2, v_3, v_4.

(Yes / No) Answer: Yes

$$\boxed{5}\,v_1 + \boxed{1}\,v_2 + \boxed{1}\,v_3 + \boxed{0}\,v_4 = 0$$

e) $\dim \text{Span}\{v_1, v_2, v_3, v_4\} = \boxed{3}$.

f) Which of the following sets form a basis for $\mathbf{R}^3$? Circle all that apply.

$\{v_1, v_2\}$ $\quad$ $\{v_1, v_2, v_3\}$ $\quad$ $\{v_1, v_3, v_4\}$

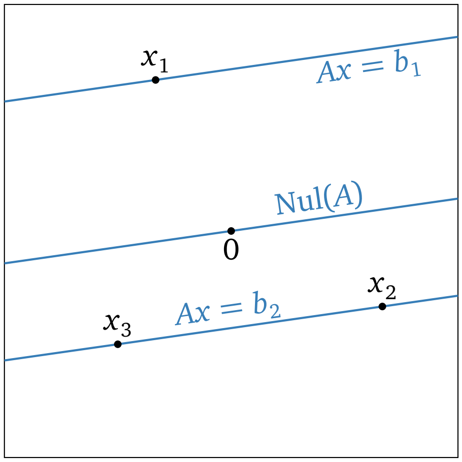
$\{v_2, v_3, v_4\}$ $\quad$ $\{v_3, v_4\}$ $\quad$ $\{v_1, v_2, v_3, v_4\}$

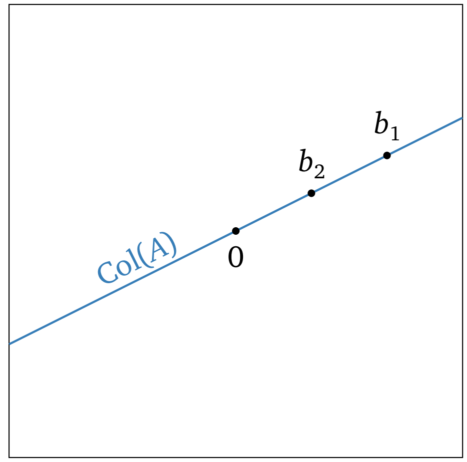
Answer (marked): $\{v_1, v_3, v_4\}$, $\{v_2, v_3, v_4\}$

Problem 4.

[10 points]

For a certain 2×2 matrix A and vectors $x_1, x_2, x_3 \in \mathbf{R}^2$ drawn on the left, the vectors $b_1 = Ax_1$ and $b_2 = Ax_2 = Ax_3$ are drawn on the right. (All vectors are drawn as points.)





In what follows, it is important that Ax_2 is equal to Ax_3.

a) Draw the solution set of $Ax = b_2$ on the picture on the left.

b) Draw the solution set of $Ax = b_1$ on the picture on the left.

c) Draw Nul(A) on the picture on the left.

d) rank(A) = $\boxed{1}$.

e) Draw Col(A) on the picture on the right.

Problem 5. [20 points]

Short-answer questions: no justification is necessary.

a) Consider the matrix

$$A = \begin{pmatrix} 3 & 7 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Which of the following are true about A? Fill in the bubbles of all that apply.

- ● A has full row rank.
- ○ A has full column rank.
- ○ A is invertible.
- ○ $\mathrm{Nul}(A) = \{0\}$.
- ● $\mathrm{Col}(A) = \mathbf{R}^3$.
- ○ There exists $b \in \mathbf{R}^3$ such that $Ax = b$ is inconsistent.
- ○ There exists $b \in \mathbf{R}^3$ such that the solution set of $Ax = b$ is a point.
- ○ The columns of A are linearly independent.
- ● The rows of A are linearly independent.

b) Which of the following are subspaces of $\mathbf{R}^3$? Fill in the bubbles of all that apply.

- ○ $\mathrm{Nul}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix}$
- ● $\mathrm{Col}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix}$
- ● $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$
- ○ The solution set of
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 2 \end{aligned}$$
- ○ $\{(x, y, z) \in \mathbf{R}^3 \ : \ xyz = 0\}$

c) Find a basis for the left null space of the matrix

$$A = \begin{pmatrix} 3 & 7 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

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d) If A is a 4×5 matrix, then

$$\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A) = \boxed{5} \qquad \dim \mathrm{Nul}(A) + \dim \mathrm{Row}(A) = \boxed{5}.$$

Problem 6. [20 points]

Find examples of the following things. If an example exists, no justification is needed; otherwise, explain why no example exists.

a) A 3×3 matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$.

No such matrix exists because $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = 3$.

b) Row-equivalent 2×2 matrices A and B with $\mathrm{Col}(A) \neq \mathrm{Col}(B)$.

There are many correct answers. For instance, $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

c) A 3×2 matrix with full row rank.

No such matrix exists, as a matrix with two columns cannot have three pivots.

d) A 2×2 matrix A such that $Ax = 0$ is inconsistent.

No such matrix exists: $Ax = 0$ always has the solution $x = 0$.