

MATH 218D-1
PRACTICE MIDTERM EXAMINATION 2

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| Name | | Duke Email | |
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may use a **3 × 5-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1.

[20 points]

a) Find an orthogonal basis for

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Now we change subspaces to avoid carry-through error. Consider the subspace

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\},$$

and note that the spanning vectors are *orthogonal*.

b) Compute the orthogonal projection of $b = (-7, 4, -4)$ onto V .

$$b_V = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix}$$

c) Compute the matrix P_V for projection onto V .

$$P_V = \frac{1}{9} \begin{pmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{pmatrix}$$

d) Compute the matrix P_{V^\perp} for projection onto V^\perp .

$$P_{V^\perp} = \frac{1}{9} \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

e) Find a basis of $\text{Nul}(P_V)$.

$$\left\{ \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \right\}$$

Problem 2.

[15 points]

In this problem we will consider the best-fit line $y = Cx + D$ through the data points

$$\begin{pmatrix} 1 \\ b_1 \end{pmatrix}, \begin{pmatrix} 2 \\ b_2 \end{pmatrix}, \begin{pmatrix} 3 \\ b_3 \end{pmatrix}, \begin{pmatrix} 4 \\ b_4 \end{pmatrix}.$$

a) The line $y = Cx + D$ passes through all four points if and only if the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

is satisfied (fill in the blank).

Let A be the coefficient matrix in the previous problem. In the QR decomposition of A , the matrix Q is

$$Q = \begin{pmatrix} 1/\sqrt{30} & 2/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{6} \\ 3/\sqrt{30} & 0 \\ 4/\sqrt{30} & -1/\sqrt{6} \end{pmatrix}.$$

b) Explain why $R = Q^T A$, and compute R .

Since $Q^T Q = I_2$, multiplying both sides of $A = QR$ by Q^T gives $Q^T A = R$.

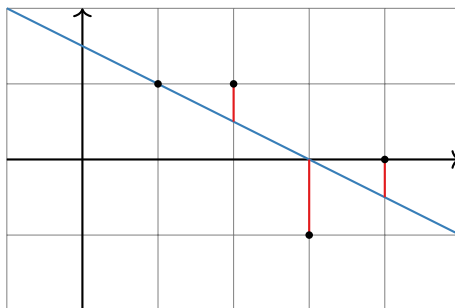
$$R = \begin{pmatrix} \sqrt{30} & \sqrt{30}/3 \\ 0 & \sqrt{6}/3 \end{pmatrix}$$

c) Use the QR decomposition to find the best-fit line through the data points

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

$$y = -\frac{1}{2}x + \frac{3}{2}$$

d) Graph the line you found in c) below. Explain which quantity was minimized in terms of the graph.



The sum of the squares of the lengths of the red lines is minimized.

Problem 3.

[20 points]

Consider the difference equation $v_{k+1} = Av_k$ where

$$A = \begin{pmatrix} 0.3 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.4 \\ 0.3 & 0.6 & 0.3 \end{pmatrix}.$$

a) Compute the characteristic polynomial $p(\lambda)$ of A .

$$p(\lambda) = -\lambda^3 + .8\lambda^2 + .2\lambda$$

b) Find the eigenvalues of A .

[Hint: one of the eigenvalues is zero, so you can factor $p(\lambda)$ using the quadratic formula.]

eigenvalues = 0, -0.2, 1

c) Find an eigenbasis $\{w_1, w_2, w_3\}$ for A .

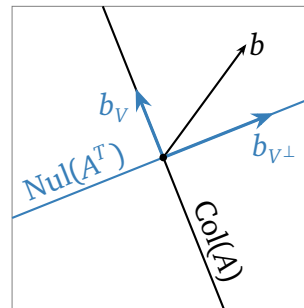
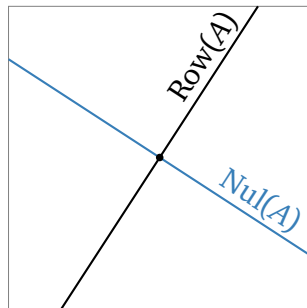
$$w_1 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad w_2 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \quad w_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

d) If $v_0 = x_1w_1 + x_2w_2 + x_3w_3$ then v_k approaches x_1w_1 as $k \rightarrow \infty$.

Problem 4.

[10 points]

For a certain 2×2 matrix A , the row space of A is drawn in the picture on the left, and the column space is drawn in the picture on the right.



- $\text{rank}(A) = 1$.
- Draw $\text{Nul}(A)$ in the picture on the left.
- Draw $\text{Nul}(A^T)$ in the picture on the right.
- If $V = \text{Col}(A)$ and b is the vector in the picture on the right, draw and label the vectors b_V and b_{V^\perp} .

Problem 5.

[20 points]

a) Find a matrix whose null space is $\text{Span}\{(1, 1, 1)\}$.

The orthogonal complement of $\text{Span}\{(1, 1, 1)\}$ is $\text{Nul}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, which has basis $\{(-1, 1, 0), (-1, 0, 1)\}$. Taking orthogonal complements again, we have

$$\text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} = \text{Nul}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

b) For which value(s) of k , if any, do the following vectors *not* form a basis of \mathbf{R}^4 ?

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -8 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ k \end{pmatrix}\right\}$$

Expanding cofactors along the first row, we compute

$$\begin{aligned} \det\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ -6 & -1 & -8 & k \end{pmatrix} &= \det\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & -8 & k \end{pmatrix} + 6 \det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \\ &= k + 32. \end{aligned}$$

Hence the columns are not linearly independent exactly when $k = -32$.

c) Which of the following properties of a matrix are not changed by row operations? (Fill in the bubbles of all that apply)

- | | |
|---|---|
| <input checked="" type="radio"/> The rank | <input type="radio"/> The left null space |
| <input type="radio"/> The column space | <input type="radio"/> The determinant |
| <input checked="" type="radio"/> The null space | <input checked="" type="radio"/> The reduced row echelon form |
| <input checked="" type="radio"/> The row space | |

d) Suppose that A is diagonalizable. Explain why A^3 is diagonalizable.

If $A = CDC^{-1}$ then $A^3 = CD^3C^{-1}$, and D^3 is still diagonal.

Problem 6.

[20 points]

Give examples of matrices with the following properties. If no such matrix exists, explain why.

- a) A 3×2 matrix A such that $Ax = (1, 2, 3)$ has more than one least-squares solution.

Any 3×2 matrix that does not have full column rank is an example.

- b) A matrix A in RREF satisfying $\dim \text{Row}(A) = 2$ and $\dim \text{Nul}(A) = 3$.

Any RREF matrix with five columns and two nonzero rows is an example.

- c) A matrix Q with orthonormal columns, such that $\det(QQ^T) = 0$.

Any non-square matrix with orthonormal columns is an example.

- d) A 2×2 matrix that is neither diagonalizable nor invertible.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$