MATH 218D-1
PRACTICE MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may use a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

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Problem 1.

[20 points]

a) Find an orthogonal basis for

$$\text{Span}\left\{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}3\\3\\-1\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\0\\0\end{pmatrix}\right\}.$$

$$\left\{ \qquad\qquad\qquad \right\}$$

[Scratch work for Problem 1]

(Problem 1, continued)

Now we change subspaces to avoid carry-through error. Consider the subspace

$$V = \text{Span}\left\{\begin{pmatrix}1\\2\\-2\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix}\right\},$$

and note that the spanning vectors are *orthogonal*.

b) Compute the orthogonal projection of $b = (-7, 4, -4)$ onto V.

$$b_V = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

c) Compute the matrix P_V for projection onto V.

$$P_V = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

d) Compute the matrix $P_{V^\perp}$ for projection onto $V^\perp$.

$$P_{V^\perp} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

e) Find a basis of $\text{Nul}(P_V)$.

$$\left\{ \qquad \right\}$$

[Scratch work for Problem 1]

Problem 2.

[15 points]

In this problem we will consider the best-fit line $y = Cx + D$ through the data points

$$\begin{pmatrix}1\\b_1\end{pmatrix}, \quad \begin{pmatrix}2\\b_2\end{pmatrix}, \quad \begin{pmatrix}3\\b_3\end{pmatrix}, \quad \begin{pmatrix}4\\b_4\end{pmatrix}.$$

a) The line $y = Cx + D$ passes through all four points if and only if the matrix equation

$$\begin{pmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}$$

is satisfied (fill in the blank).

Let A be the coefficient matrix in the previous problem. In the QR decomposition of A, the matrix Q is

$$Q = \begin{pmatrix} 1/\sqrt{30} & 2/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{6} \\ 3/\sqrt{30} & 0 \\ 4/\sqrt{30} & -1/\sqrt{6} \end{pmatrix}.$$

b) Explain why $R = Q^T A$, and compute R.

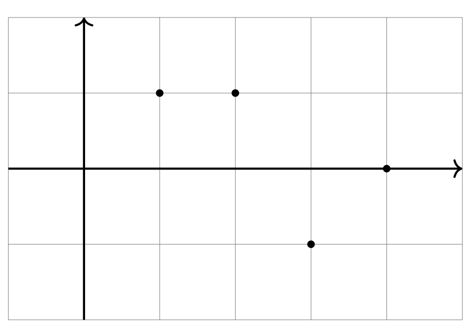
$$R = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

c) Use the QR decomposition to find the best-fit line through the data points

$$\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}3\\-1\end{pmatrix}, \quad \begin{pmatrix}4\\0\end{pmatrix}.$$

$$y = \boxed{}\, x + \boxed{}$$

d) Graph the line you found in **c)** below. Explain which quantity was minimized in terms of the graph.



[Scratch work for Problem 2]

Problem 3.

[20 points]

Consider the difference equation $v_{k+1} = Av_k$ where

$$A = \begin{pmatrix} 0.3 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.4 \\ 0.3 & 0.6 & 0.3 \end{pmatrix}.$$

a) Compute the characteristic polynomial $p(\lambda)$ of A.

$$p(\lambda) =$$

b) Find the eigenvalues of A.

[**Hint:** one of the eigenvalues is zero, so you can factor $p(\lambda)$ using the quadratic formula.]

eigenvalues $= 0,$

c) Find an eigenbasis $\{w_1, w_2, w_3\}$ for A.

$$w_1 = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \quad w_2 = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \quad w_3 = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

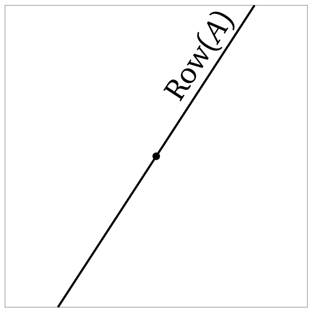
d) If $v_0 = x_1w_1 + x_2w_2 + x_3w_3$ then v_k approaches $\boxed{}$ as $k \to \infty$.

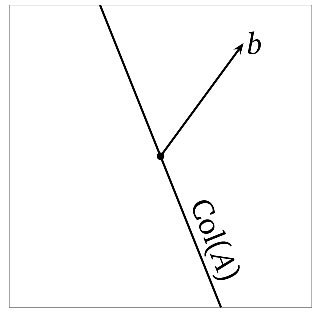
[Scratch work for Problem 3]

Problem 4.

[10 points]

For a certain 2×2 matrix A, the row space of A is drawn in the picture on the left, and the column space is drawn in the picture on the right.





a) $\text{rank}(A) = \square$.

b) Draw $\text{Nul}(A)$ in the picture on the left.

c) Draw $\text{Nul}(A^T)$ in the picture on the right.

d) If $V = \text{Col}(A)$ and b is the vector in the picture on the right, draw and label the vectors b_V and $b_{V^\perp}$.

[Scratch work for Problem 4]

Problem 5.

[20 points]

a) Find a matrix whose null space is $\text{Span}\{(1,1,1)\}$.

b) For which value(s) of k, if any, do the following vectors *not* form a basis of $\mathbf{R}^4$?

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -8 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ k \end{pmatrix} \right\}$$

c) Which of the following properties of a matrix are not changed by row operations? (Fill in the bubbles of all that apply)

- ◯ The rank
- ◯ The column space
- ◯ The null space
- ◯ The row space
- ◯ The left null space
- ◯ The determinant
- ◯ The reduced row echelon form

d) Suppose that A is diagonalizable. Explain why A^3 is diagonalizable.

[Scratch work for Problem 5]

Problem 6. [20 points]

Give examples of matrices with the following properties. If no such matrix exists, explain why.

a) A 3×2 matrix A such that $Ax = (1, 2, 3)$ has more than one least-squares solution.

b) A matrix A in RREF satisfying $\dim \mathrm{Row}(A) = 2$ and $\dim \mathrm{Nul}(A) = 3$.

c) A matrix Q with orthonormal columns, such that $\det(QQ^T) = 0$.

d) A 2×2 matrix that is neither diagonalizable nor invertible.

[Scratch work for Problem 6]