MATH 218D-1
MIDTERM EXAMINATION 2

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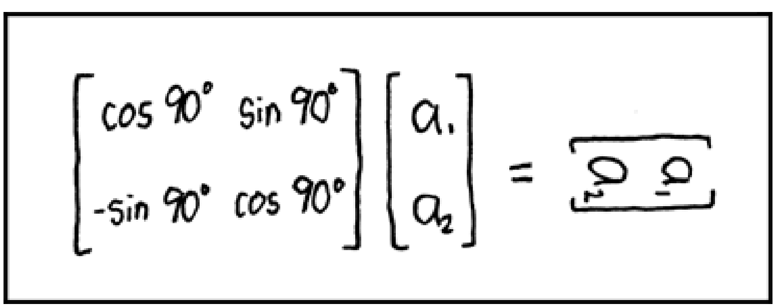
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3 × 5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



[Hint: this is a joke.]

Problem 1. [20 points]

Consider the subspace V of $\mathbf{R}^4$ defined by the equation

$$x_1 - x_2 + 2x_3 - 6x_4 = 0.$$

a) Compute an *orthogonal* basis for V.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

b) Compute an *orthogonal* basis for $V^\perp$.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ -6 \end{pmatrix} \right\}$$

c) Compute the projection matrix P_V.

$$P_V = \frac{1}{42} \begin{pmatrix} 41 & 1 & -2 & 6 \\ 1 & 41 & 2 & -6 \\ -2 & 2 & 38 & 12 \\ 6 & -6 & 12 & 6 \end{pmatrix}$$

d) Compute the orthogonal projection of the vector $b = (1, 0, 1, -3)$ onto V.

$$b_V = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

e) The distance from $(1, 0, 1, -3)$ to V is $\boxed{\sqrt{21}/\sqrt{2}}$.

Problem 2. [15 points]

Applying the Gram–Schmidt procedure to a certain list of vectors v_1, v_2, v_3 in $\mathbf{R}^4$ yields the vectors

$$\begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} = u_1 = v_1 \qquad \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} = u_2 = v_2 + 2u_1 \qquad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} = u_3 = v_3 - \frac{3}{2}u_1 + \frac{1}{2}u_2.$$

The following questions are easier if you do not compute v_2 and v_3.

a) $\dfrac{v_1 \cdot v_2}{v_1 \cdot v_1} = \boxed{-2}$

b) What is the orthogonal projection of v_3 onto $V_2 = \mathrm{Span}\{u_1, u_2\}$?

$$(v_3)_{V_2} = \boxed{3/2}\, u_1 + \boxed{-1/2}\, u_2$$

c) What is the orthogonal projection of $b = (0, 5, -5, 0)$ onto $V = \mathrm{Span}\{v_1, v_2, v_3\}$?

$$b_V = \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}$$

d) Let A be the matrix with columns v_1, v_2, v_3. The QR decomposition of A is

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{5} & -4\sqrt{5} & 3\sqrt{5} \\ 0 & 2\sqrt{5} & -\sqrt{5} \\ 0 & 0 & 2\sqrt{5} \end{pmatrix}$$

e) The least-squares solution of $A\widehat{x} = b$ (with A and b as above) is

$$\widehat{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Problem 3.

[15 points]

a) Compute the characteristic polynomial of the matrix

$$\begin{pmatrix} 2 & 3 & -6 \\ -6 & -7 & 12 \\ -3 & -3 & 5 \end{pmatrix}.$$

Do not factor your answer.

$$p(\lambda) = -\lambda^3 + 3\lambda + 2$$

Now we switch matrices to avoid carry-through error. The matrix

$$A = \begin{pmatrix} -7 & -18 & 30 \\ -12 & -37 & 60 \\ -9 & -27 & 44 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = -(\lambda+1)^2(\lambda-2)$.

b) The eigenvalues of A are $\lambda_1 = \boxed{-1}$ and $\lambda_2 = \boxed{2}$.

c) Compute a basis for each eigenspace. Scale your eigenvectors to have integer (whole-number) entries.

$$\lambda_1 : \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \lambda_2 : \left\{ \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \right\}$$

d) Solve the difference equation

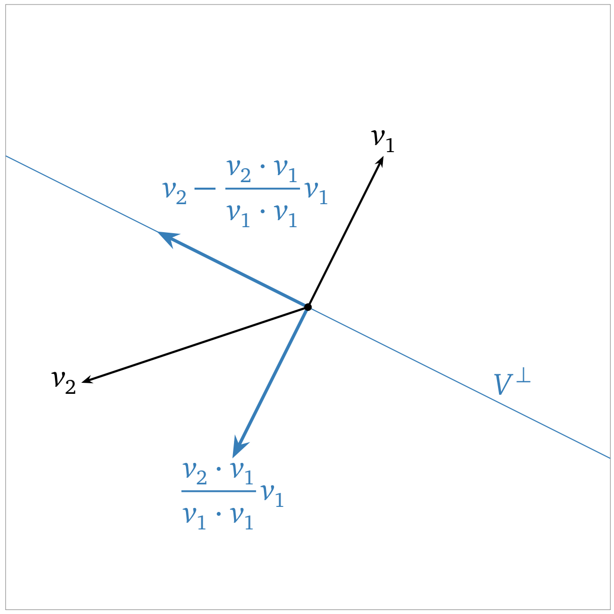
$$v_{k+1} = Av_k \qquad v_0 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

$$v_k = \begin{pmatrix} -2(-1)^k + 2 \cdot 2^k \\ -(-1)^k + 4 \cdot 2^k \\ -(-1)^k + 3 \cdot 2^k \end{pmatrix}$$

Problem 4.

[10 points]

Certain vectors v_1 and v_2 are drawn below.



Draw and label:

a) $\frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$ **b)** $v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$

c) The orthogonal complement of $V = \mathrm{Span}\{v_1\}$.

Problem 5. [20 points]

a) Let A be an $m \times n$ matrix and let $b \in \mathbf{R}^n$ be a vector. Explain why b can be expressed as a sum of a vector in Row(A) and a vector in Nul(A).

Let $V = \text{Row}(A)$. Then $V^{\perp} = \text{Nul}(A)$, and $b_V + b_{V^{\perp}} = b$.

b) Performing the following sequence of row operations on a matrix A results in a matrix U in reduced row echelon form:

$$A \xrightarrow{R_1 \mathrel{+}= 2R_2,\ R_2 \mathrel{\times}= 3,\ R_1 \mathrel{-}= R_3,\ R_2 \longleftrightarrow R_3} U = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

What is det(A)?

$\det(A) = 0$

c) Consider the subspace

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 7 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \\ -1 \end{pmatrix} \right\}$$

and the projection matrix P_V. There exists an invertible matrix C such that $P_V = CDC^{-1}$, where D is the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

d) Suppose that λ is an eigenvalue of A. Which of the following statements can you conclude? Fill in the circles of all that apply.

- ● $A - \lambda I_n$ has a free variable.
- ● There exists a vector $v \in \mathbf{R}^n$ such that $Av = \lambda v$.
- ● λ^2 is an eigenvalue of A^2.
- ○ $A = CDC^{-1}$ for an invertible matrix C and a diagonal matrix D.
- ● 0 is an eigenvalue of $A - \lambda I_n$.
- ● λ is a zero of the characteristic polynomial of A.

Problem 6. [20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A diagonalizable 2×2 matrix with characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$.

There are many answers. One is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

b) An invertible 2×2 matrix with characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$.

This is not possible: $\det(A) = p(0) = 0$.

c) A matrix A such that $b_V = b$, where $b = (1, 2, 1)$ and $V = \mathrm{Col}(A)$.

Any matrix with b as a column will work.

d) A 2×2 symmetric matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$.

This is not possible: if A is symmetric then $\mathrm{Col}(A) = \mathrm{Row}(A) = \mathrm{Nul}(A)^{\perp}$.

e) A 2×2 matrix with no (real) eigenvectors.

There are many answers. One is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$