

**MATH 218D-1**  
**MIDTERM EXAMINATION 2**

<b>Name</b>		<b>Duke Email</b>	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a **3 × 5-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[Hint: this is a joke.]

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## Problem 1.

[20 points]

Consider the subspace  $V$  of  $\mathbf{R}^4$  defined by the equation

$$x_1 - x_2 + 2x_3 - 6x_4 = 0.$$

a) Compute an *orthogonal* basis for  $V$ .

{ }

b) Compute an *orthogonal* basis for  $V^\perp$ .

{ }

[Scratch work for Problem 1]

(Problem 1, continued)

c) Compute the projection matrix  $P_V$ .

$$P_V = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

d) Compute the orthogonal projection of the vector  $b = (1, 0, 1, -3)$  onto  $V$ .

$$b_V = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

e) The distance from  $(1, 0, 1, -3)$  to  $V$  is .

[Scratch work for Problem 1]

## Problem 2.

[15 points]

Applying the Gram–Schmidt procedure to a certain list of vectors  $v_1, v_2, v_3$  in  $\mathbf{R}^4$  yields the vectors

$$\begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} = u_1 = v_1 \quad \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} = u_2 = v_2 + 2u_1 \quad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} = u_3 = v_3 - \frac{3}{2}u_1 + \frac{1}{2}u_2.$$

The following questions are easier if you do not compute  $v_2$  and  $v_3$ .

a)  $\frac{v_1 \cdot v_2}{v_1 \cdot v_1} = \boxed{\phantom{00}}$

b) What is the orthogonal projection of  $v_3$  onto  $V_2 = \text{Span}\{u_1, u_2\}$ ?

$$(v_3)_{V_2} = \boxed{\phantom{00}}u_1 + \boxed{\phantom{00}}u_2$$

c) What is the orthogonal projection of  $b = (0, 5, -5, 0)$  onto  $V = \text{Span}\{v_1, v_2, v_3\}$ ?

$$b_V = \begin{pmatrix} \phantom{00} \\ \phantom{00} \\ \phantom{00} \\ \phantom{00} \end{pmatrix}$$

[Scratch work for Problem 2]



(Problem 2, continued)

d) Let  $A$  be the matrix with columns  $v_1, v_2, v_3$ . The QR decomposition of  $A$  is

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

e) The least-squares solution of  $A\hat{x} = b$  (with  $A$  and  $b$  as above) is

$$\hat{x} = \begin{pmatrix} \\ \\ \end{pmatrix}.$$

[Scratch work for Problem 2]

### Problem 3.

[15 points]

- a) Compute the characteristic polynomial of the matrix

$$\begin{pmatrix} 2 & 3 & -6 \\ -6 & -7 & 12 \\ -3 & -3 & 5 \end{pmatrix}.$$

Do not factor your answer.

$$p(\lambda) =$$

[Scratch work for Problem 3]

**(Problem 3, continued)**

Now we switch matrices to avoid carry-through error. The matrix

$$A = \begin{pmatrix} -7 & -18 & 30 \\ -12 & -37 & 60 \\ -9 & -27 & 44 \end{pmatrix}$$

has characteristic polynomial  $p(\lambda) = -(\lambda + 1)^2(\lambda - 2)$ .

b) The eigenvalues of  $A$  are  $\lambda_1 = \boxed{\phantom{00}}$  and  $\lambda_2 = \boxed{\phantom{00}}$ .

c) Compute a basis for each eigenspace. Scale your eigenvectors to have integer (whole-number) entries.

$$\lambda_1: \left\{ \begin{array}{c} \phantom{00} \\ \phantom{00} \\ \phantom{00} \end{array} \right\} \quad \lambda_2: \left\{ \begin{array}{c} \phantom{00} \\ \phantom{00} \\ \phantom{00} \end{array} \right\}$$

d) Solve the difference equation

$$v_{k+1} = Av_k \quad v_0 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

$$v_k = \begin{pmatrix} \phantom{00} \\ \phantom{00} \\ \phantom{00} \end{pmatrix}$$

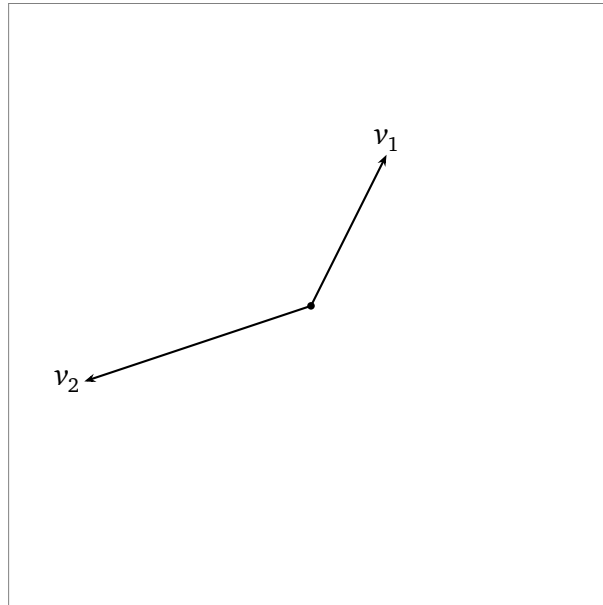
(the entries of  $v_k$  should be expressions containing  $k$ ).

[Scratch work for Problem 3]

### Problem 4.

[10 points]

Certain vectors  $v_1$  and  $v_2$  are drawn below.



Draw and label:

**a)**  $\frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$       **b)**  $v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$

**c)** The orthogonal complement of  $V = \text{Span}\{v_1\}$ .

[Scratch work for Problem 4]



## Problem 5.

[20 points]

- a) Let  $A$  be an  $m \times n$  matrix and let  $b \in \mathbf{R}^n$  be a vector. Explain why  $b$  can be expressed as a sum of a vector in  $\text{Row}(A)$  and a vector in  $\text{Nul}(A)$ .

- b) Performing the following sequence of row operations on a matrix  $A$  results in a matrix  $U$  in reduced row echelon form:

$$A \xrightarrow{R_1 += 2R_2, R_2 \times= 3, R_1 -= R_3, R_2 \longleftrightarrow R_3} U = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

What is  $\det(A)$ ?

- c) Consider the subspace

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 7 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 3 \\ -1 \end{pmatrix} \right\}$$

and the projection matrix  $P_V$ . There exists an invertible matrix  $C$  such that  $P_V = CDC^{-1}$ , where  $D$  is the diagonal matrix

$$D = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}.$$

[Hint: no computations are necessary.]

- d) Suppose that  $\lambda$  is an eigenvalue of  $A$ . Which of the following statements can you conclude? Fill in the circles of all that apply.
- $A - \lambda I_n$  has a free variable.
  - There exists a vector  $v \in \mathbf{R}^n$  such that  $Av = \lambda v$ .
  - $\lambda^2$  is an eigenvalue of  $A^2$ .
  - $A = CDC^{-1}$  for an invertible matrix  $C$  and a diagonal matrix  $D$ .
  - $0$  is an eigenvalue of  $A - \lambda I_n$ .
  - $\lambda$  is a zero of the characteristic polynomial of  $A$ .

[Scratch work for Problem 5]

## Problem 6.

[20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A diagonalizable  $2 \times 2$  matrix with characteristic polynomial  $p(\lambda) = \lambda^2 - \lambda$ .

b) An invertible  $2 \times 2$  matrix with characteristic polynomial  $p(\lambda) = \lambda^2 - \lambda$ .

c) A matrix  $A$  such that  $b_V = b$ , where  $b = (1, 2, 1)$  and  $V = \text{Col}(A)$ .

d) A  $2 \times 2$  symmetric matrix  $A$  such that  $\text{Col}(A) = \text{Nul}(A)$ .

e) A  $2 \times 2$  matrix with no (real) eigenvectors.

[Scratch work for Problem 6]