Math 218D-1: Homework #1

due Wednesday, January 18, at 11:59pm

1. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u+v+w$ and $u+2v-w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Explain why every linear combination of u, v, w is also a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to ____?

e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

2. Find two *different* triples (x, y, z) such that

$$x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

How many such triples are there?

3. Decide if each statement is true or false, and explain why.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

c) If v, w are two vectors in $\mathbf{R}^2$, then any other vector b in $\mathbf{R}^2$ is a linear combination of v and w.

4. Suppose that v and w are *unit vectors*: that is, $v \cdot v = 1$ and $w \cdot w = 1$. Compute the following dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v+w)\cdot(v-w)$ **c)** $(v+2w)\cdot(v-2w)$.

5. Two vectors v and w are *orthogonal* if $v \cdot w = 0$, and they are *parallel* if one is a scalar multiple of the other. A *unit vector* is a vector v with $v \cdot v = 1$.

Decide if each statement is true or false, and explain why.

a) If $u = (1, 1, 1)$ is orthogonal to v and to w, then v is parallel to w.

b) If u is orthogonal to $v + w$ and to $v - w$, then u is orthogonal to v and w.

c) If u and v are orthogonal unit vectors then $(u-v)\cdot(u-v) = 2$.

d) If $u \cdot u + v \cdot v = (u+v)\cdot(u+v)$, then u and v are orthogonal.

6. Find nonzero vectors v and w that are orthogonal to $(1,1,1)$ and to each other.

7. Compute the following matrix-vector products using *both* the by-row and by-column methods. If the product is not defined, explain why.

$$\begin{pmatrix}2\\5\end{pmatrix}\begin{pmatrix}1\\-3\\-1\end{pmatrix} \qquad \begin{pmatrix}1&-2\\0&-1\\3&2\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix} \qquad \begin{pmatrix}7&2&4\\3&-3&1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}7&4\\-2&2\\4&1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} \qquad \begin{pmatrix}2&6&-1\end{pmatrix}\begin{pmatrix}5\\-1\\0\end{pmatrix} \qquad \begin{pmatrix}5\\-1\\0\end{pmatrix}\begin{pmatrix}2&6&-1\end{pmatrix}$$

8. Suppose that $u = (x,y,z)$ and $v = (a,b,c)$ are vectors satisfying $2u + 3v = 0$. Find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix}x&a\\y&b\\z&c\end{pmatrix}w = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

9. Consider the matrices

$$A = \begin{pmatrix}2&1&-1\\4&-4&2\end{pmatrix} \qquad B = \begin{pmatrix}5&3&2\\1&-1&2\end{pmatrix} \qquad C = \begin{pmatrix}1&2\\2&-1\end{pmatrix}$$

$$D = \begin{pmatrix}3&1\\-1&2\end{pmatrix} \qquad E = \begin{pmatrix}-3&5\end{pmatrix}.$$

Compute the following expressions. If the result is not defined, explain why.

a) $-3A$ **b)** $B - 3A$ **c)** AC **d)** B^2

e) $A + 2B$ **f)** $C - E$ **g)** EB **h)** D^2

10. Compute the product

$$\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}2&1&-1\\4&-1&2\end{pmatrix}$$

in three ways:

a) Using the column form and the “by columns” method on each column.

b) Using the column form and the “by rows” method on each column.

c) Using the outer product form.

11. Consider the matrices

$$A = \begin{pmatrix}1&2\\-2&1\end{pmatrix} \qquad B = \begin{pmatrix}1&1\\-1&h\end{pmatrix}.$$

What value(s) of h, if any, will make $AB = BA$?

12. Consider the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} -4 & -8 \\ 5 & 8 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Verify that $AC = BC$ and yet $A \neq B$.

13. Show that $(A+B)^2 \neq A^2 + 2AB + B^2$ when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

What is the correct formula?

$$(A+B)^2 = A^2 + B^2 + \underline{\qquad\quad}$$

14. In the following, find the 2×2 matrix A that acts in the specified manner.

a) $A\binom{x}{y} = \binom{x}{y}$: the identity matrix does not change the vector.

b) $A\binom{x}{y} = \binom{y}{x}$: this is a flip over the line $y = x$.

c) $A\binom{x}{y} = \binom{y}{-x}$: this rotates vectors counterclockwise by 90°.

d) $A\binom{x}{y} = -\binom{x}{y}$: this rotates vectors by 180°.

e) $A\binom{x}{y} = \binom{0}{y}$: this projects onto the y-axis.

f) $A\binom{x}{y} = \binom{x}{0}$: this projects onto the x-axis.

g) $A\binom{x}{y} = \binom{x}{y-2x}$: this performs the row operation $R_2 -= 2R_1$.

15. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by the diagonal matrix D on the right or the left.

16. Let A be a 4×3 matrix satisfying

$$Ae_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 7 \end{pmatrix} \qquad Ae_2 = \begin{pmatrix} 4 \\ 4 \\ -1 \\ -1 \end{pmatrix} \qquad Ae_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

What is A?

17. Suppose that A is a 4×3 matrix such that

$$A\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}7\\1\\2\\9\end{pmatrix} \qquad A\begin{pmatrix}0\\1\\4\end{pmatrix} = \begin{pmatrix}1\\-4\\2\\3\end{pmatrix}.$$

Compute Ax, where x is the vector $(3,-2,-2)$.

18. For the following matrices A and B, compute AB, A^T, B^T, B^TA^T, and $(AB)^T$. Which of these matrices are equal and why? Why can't you compute A^TB^T?

$$A = \begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix} \qquad B = \begin{pmatrix}2 & 1 & -1\\4 & -1 & 2\end{pmatrix}.$$

19. Decide if each statement is true or false, and explain why.

a) If A and B are symmetric of the same size, then AB is symmetric.

b) If A is symmetric, then A^3 is symmetric.

c) If A is any matrix, then A^TA is symmetric.