

Math 218D-1: Homework #11
due Wednesday, April 5, at 11:59pm

1. Compute the following complex numbers.

a) $(1+i) + (2-i)$ b) $(1+i)(2-i)$ c) $\overline{2-i}$ d) $\frac{1+i}{2-i}$
e) $|1+i|$ f) $2e^{2\pi i/3}$ g) $5e^{3\pi i}$

2. Express each complex number in polar coordinates $re^{i\theta}$.

a) $1+i$ b) $\frac{-1+i\sqrt{3}}{2}$ c) $-\sqrt{3}-3i$ d) $\frac{1}{1+i}$ e) $(1-i\sqrt{3})^n$

3. For which numbers θ is $e^{i\theta} = 1$? What about -1 ?

4. For each matrix A and each vector x , decide if x is an eigenvector of A , and if so, find the eigenvalue λ .

a) $\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}$ b) $\begin{pmatrix} -4 & 13 & 13 \\ 2 & -2 & -4 \\ -4 & 8 & 10 \end{pmatrix}, \begin{pmatrix} 1+5i \\ -2i \\ 4i \end{pmatrix}$
c) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ -2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2+i \\ 1 \\ -i \end{pmatrix}$

Careful! It is difficult to recognize by inspection if two complex vectors are (complex) scalar multiples of each other.

5. For each 2×2 matrix A , **i)** compute the characteristic polynomial, **ii)** find all (real and complex) eigenvalues, and **iii)** find a basis for each eigenspace, using HW9#13 when applicable. **iv)** Is the matrix diagonalizable (over the complex numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

a) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ c) $\begin{pmatrix} -3 & 5 \\ -10 & 7 \end{pmatrix}$

6. Diagonalize the following matrix over the complex numbers:¹

$$A = \begin{pmatrix} 1 & 4 & -6 \\ -6 & 7 & -22 \\ -2 & 1 & -5 \end{pmatrix}.$$

¹This problem is included to make you do Gaussian elimination by hand with complex numbers *one time*, so that you'll be grateful to have computers do it for you in the future.

7. A certain forest contains a population of rabbits and a population of foxes. If there are r_n rabbits and f_n foxes in year n , then

$$\begin{aligned} r_{n+1} &= 3r_n - f_n \\ f_{n+1} &= r_n + 2f_n \end{aligned}$$

in other words, each rabbit produces three baby rabbits on average, but there is some loss due to predation by foxes; each fox produces two babies on average, but this is increased with ample prey.

- a) Let $v_n = \begin{pmatrix} r_n \\ f_n \end{pmatrix}$. Find a matrix A such that $v_{n+1} = Av_n$.
- b) Find an eigenbasis of A . (The eigenvectors and eigenvalues will be complex.)
[Hint: Part d) will be easier if you choose the eigenvectors with first coordinate equal to 1.]
- c) Suppose that $r_0 = 2$ and $f_0 = 1$. Find closed formulas for r_n and f_n . Find a formula for r_n involving only real numbers. (This latter formula can involve an arctan.)
- d) In this model, the populations do not stabilize. How many years will it take for the foxes to eat all of the rabbits?

In general, any 2×2 difference equation with a complex eigenvalue will exhibit oscillation centered at zero. This phenomenon can be described explicitly, but is beyond the scope of this course.

8.
 - a) Let A be an $n \times n$ matrix. Prove that λ is an eigenvalue of A with geometric multiplicity n if and only if $A = \lambda I_n$.
 - b) Find a non-diagonal 2×2 matrix such that 1 is an eigenvalue with algebraic multiplicity 2.
9. Find examples of real 2×2 matrices A with the following properties.
 - a) A is invertible and diagonalizable over the real numbers.
 - b) A is invertible but not diagonalizable over the complex numbers.
 - c) A is diagonalizable over the real numbers but not invertible.
 - d) A is neither invertible nor diagonalizable over the complex numbers.

This shows that *invertibility and diagonalizability have nothing to do with each other*.

10. Let A be an $n \times n$ matrix.
 - a) Show that the product of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\det(A)$.
 - b) [Optional] Show that the sum of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\text{Tr}(A)$.

(Both of these are identities involving the characteristic polynomial of A .)

11. Let V be a plane in \mathbf{R}^3 , let $L = V^\perp$ be the orthogonal line, let P_L be the matrix for orthogonal projection onto L , and let $R_V = I_3 - 2P_L$ be the reflection over V , as in HW9#5.

a) Prove that there exists an invertible 3×3 matrix C such that

$$P_L = C \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} C^{-1}.$$

Use this to show that the characteristic polynomial of P_L is $-\lambda^2(\lambda - 1)$.

b) Prove that

$$R_V = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} C^{-1}$$

for the same matrix C of part a). Use this to show that the characteristic polynomial of R_V is $-(\lambda - 1)^2(\lambda + 1)$ and that $\det(R_V) = -1$.

(Compare HW9#5, HW10#3, and HW10#11.)

12. For each matrix in HW10#2(a)–(c), compute the algebraic and geometric multiplicity of each eigenvalue. What does your answer say about diagonalizability?

Optional: do (d)–(g) as well.

13. Give an example of each of the following, or explain why no such example exists. All matrices should have real entries.

a) A 3×3 matrix with eigenvalues 0, 1, 2, and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

b) A 4×4 matrix having eigenvalue 2 with algebraic multiplicity 2 and geometric multiplicity 3.

c) A 3×3 matrix with one complex (non-real) eigenvalue and two real eigenvalues.

d) A 2×2 matrix A such that A^2 is diagonalizable over the real numbers but A is not diagonalizable, even over the complex numbers.

[**Hint:** try a nonzero matrix A such that $A^2 = 0$.]

- 14.** Decide if each statement is true or false, and explain why.
- a) If A and B are diagonalizable $n \times n$ matrices, then so is AB .
 - b) An $n \times n$ matrix with n (different) eigenvalues is diagonalizable.
 - c) An $n \times n$ matrix is diagonalizable if it has n eigenvalues, counted with algebraic multiplicity.
 - d) Any 2×2 real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.
 - e) Any 3×3 real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.
 - f) Any 4×4 real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.
 - g) Any 2×2 real matrix has a real eigenvalue.
 - h) Any 3×3 real matrix has a real eigenvalue.
 - i) Any $n \times n$ matrix has a (real or complex) eigenvalue.
 - j) If the characteristic polynomial of A is $-(\lambda^3 - 1) = -(\lambda^2 + \lambda + 1)(\lambda - 1)$, then the 1-eigenspace of A is a line.

- 15.** Solve the following initial value problems. Your solutions should involve only real numbers.

$$\text{a) } \begin{cases} u_1' = u_1 - 2u_2 & u_1(0) = -3 \\ u_2' = u_1 + 4u_2 & u_2(0) = 2 \end{cases} \quad \text{b) } \begin{cases} u_1' = 3u_1 - u_2 & u_1(0) = 4 \\ u_2' = u_1 + 2u_2 & u_2(0) = 2 \end{cases}$$

- 16.** Solve the following initial value problem.

$$p''(t) = -2p(t) + 3p'(t) \quad p(0) = 1 \quad p'(0) = -1.$$

- 17.** Use the matrix exponential to solve the following initial value problem.

$$\begin{cases} u_1' = 2u_2 + u_3 & u_1(0) = 2 \\ u_2' = -u_3 & u_2(0) = 3 \\ u_3' = 0 & u_3(0) = -1. \end{cases}$$

(This is one of the few instances where the matrix exponential leads to a computable solution!)