Math 218D-1: Homework #13

due Wednesday, April 19, at 11:59pm

1. For each quadratic form $q(x_1,x_2)$ of HW12#15**(a,b)**, first **i)** draw the solutions of $q(x_1,x_2)=1$, being sure to draw the shortest and longest solutions, and then **ii)** find the maximum and minimum values of $\|x\|^2$ subject to the constraint $q(x)=1$, and at which points (x_1,x_2) these values are attained.

What happens if you try to extremize $\|x\|^2$ subject to

$$q(x_1,x_2)=x_1^2-6x_1x_2+x_2^2=1?$$

(This is the form from part **(c)** of HW12#15.)

2. **a)** Consider the quadratic form

$$q(x_1,x_2,x_3)=7x_1^2+6x_2^2+5x_3^2+4x_1x_2+4x_2x_3,$$

of HW12#16. Find the *smallest* value of $q(x)$ subject to the constraints $\|x\|=1$ *and* $x\perp\frac{1}{3}(1,-2,2)$. At which vectors x is this minimum attained?

b) Consider the quadratic form

$$q(x_1,x_2,x_3)=x_1^2+x_2^2+7x_3^2-16x_1x_2+8x_1x_3+8x_2x_3.$$

of HW12#17. Find the *largest* value of $q(x)$ subject to the constraints $\|x\|=1$ *and* $x\perp\frac{1}{\sqrt{5}}(0,1,2)$. At which vectors x is this maximum attained?

3. For each matrix A, find the minimum and maximum values of $\|Ax\|^2$ subject to the constraint $\|x\|=1$. At which vectors are these extrema achieved? Check your work by choosing a unit vector x maximizing $\|Ax\|^2$, computing $b=Ax$, and verifying that $\|b\|^2$ is equal to the maximum.

a) $\begin{pmatrix}3&-1\\2&3\\1&-1\end{pmatrix}$ **b)** $\begin{pmatrix}1&-3\\0&0\end{pmatrix}$ **c)** $\begin{pmatrix}2&0&-1\\1&1&2\\1&-1&0\\1&2&0\end{pmatrix}$

4. Consider the matrix

$$A=\begin{pmatrix}3&2&-1&4&-3\\1&7&-2&3&-5\\2&0&8&-1&1\\1&2&0&3&9\end{pmatrix}.$$

a) Find a unit vector u_1 maximizing $\|Ax\|^2$ subject to $\|x\|=1$.

b) Find the maximum value of $\|Ax\|^2$ subject to $\|x\|=1$ and $x\perp u_1$.

c) Find the minimum value of $\|Ax\|$ subject to $\|x\|=1$ without doing any work.

You'll need to use a computer algebra system. With the Sage cell on the course webpage, you'd want something like this:

```
A = Matrix([[ 3., 2.,-1., 4.,-3.],
            [ 1., 7.,-2., 3.,-5.],
            [ 2., 0., 8.,-1., 1.],
            [ 1., 2., 0., 3., 9.]])
pprint((A.transpose()*A).eigenvects())
```

(Entering numbers as "`3.`" instead of "3" forces SymPy to perform a floating-point computation instead of a symbolic one.)

5. In this problem, we will touch on the role of quadratic optimization in *spectral graph theory*. Spectral graph theory is the study of graphs using linear algebra, and is widely applied to problems in networking and partitioning. (Google's PageRank algorithm can be formulated as a spectral graph theory problem.)

A *graph* is a set of *vertices*, or points, connected by a set of *edges*. For simplicity, we will assume that each edge has distinct endpoints (i.e., there are no loop edges), and that there is at most one edge connecting any two vertices: such a graph is called *simple*. Under these assumptions, an edge is determined by the two vertices it connects, so we can write $e = (1,2)$ for the edge connecting vertices 1 and 2. We also write $i \sim j$ if (i, j) is an edge of G. The *degree* of a vertex is the number of edges connected to it; the degree of vertex i is written $\deg(i)$.

Let G be a graph with n vertices labeled $1, 2, \ldots, n$. We consider a vector $x \in \mathbf{R}^n$ as a way to assign a real number to each vertex: the ith coordinate x_i is the number attached to the ith vertex. The *Laplacian* of G is the $n \times n$ matrix L whose (i, j) entry is

$$L_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if there is an edge from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise.} \end{cases}$$

Note that L is symmetric. Let $x \in \mathbf{R}^n$ and let $y = Lx$. Then the ith coordinate of y is

$$(\star) \qquad y_i = x_i \deg(i) - \sum_{j \sim i} x_j = \sum_{j \sim i} (x_i - x_j).$$

In other words, y is the vector that assigns the number $\sum_{j \sim i}(x_i - x_j)$ to vertex i.

The eigenvalues of the graph Laplacian contain important information about the structure of the graph.

a) Show that the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{R}^n$ is in the null space of L.

It follows that 0 is always an eigenvalue of L.

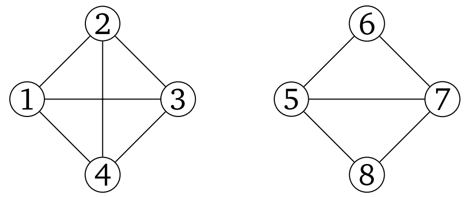
b) Show that $x^T L x = \sum_{j \sim i} (x_i - x_j)^2$. Explain why L is positive-semidefinite.

Since L is positive-semidefinite, all of its eigenvalues are *nonnegative*, so 0 is the smallest eigenvalue of L. The fact that 0 is an eigenvalue gives us no information about the graph, so we wish to "rule it out" by imposing the constraint $x \perp \mathbf{1}$.

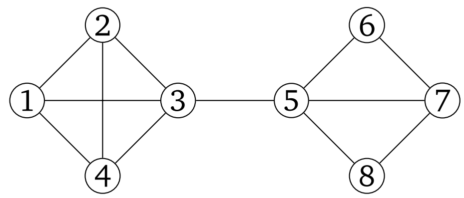
According to **b)**, minimizing $q(x) = x^T L x$ subject to the constraints $\|x\|^2 = 1$ and $x \perp \mathbf{1}$ amounts to finding a way to assign a number to each vertex such that *neighboring vertices have similar values*, but such that the sum of the values is zero ($x \perp \mathbf{1}$) and the sum of their squares is 1 ($\|x\| = 1$).

For each of the following graphs, **i)** compute the Laplacian matrix L and **ii)** minimize $x^T L x$ subject to $x \perp \mathbf{1}$ and $\|x\| = 1$. **iii)** For a (unit) vector x achieving this minimum, draw the number x_i next to vertex i on the graph. **iv)** What does the second-smallest eigenvalue say about the graph? (This is open-ended.)

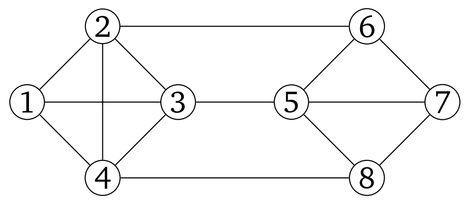
c)



d)



e)



You should feel free to use a computer algebra system to compute the eigenvalues and eigenvectors. For instance, you can use SymPy in the Sage cell on the course webpage. Finding the eigenvalues and eigenvectors of a matrix in SymPy is done as follows: if your matrix is

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

then you would type:

```
A = Matrix([[7.,2.,0.],[2.,6.,2.],[0.,2.,5.]])
pprint(A.eigenvects())
```

(Entering numbers as "`3.`" instead of "3" forces SymPy to perform a floating-point computation instead of a symbolic one.) The output is a list of tuples of the form (eigenvalue, multiplicity, eigenspace basis)—note that the eigenspace basis will not necessarily be orthonormal.

6. For each matrix A, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

$$\textbf{a)} \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} \qquad \textbf{e)} \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

7. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 6(a). Let σ_1, σ_2 be the singular values of A. Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

8. Let A be a matrix with nonzero orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of A in outer product form.

9. Let S be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal eigenbasis, where v_i has eigenvalue λ_i.

a) Show that the singular values of S are $|\lambda_1|, \ldots, |\lambda_r|$. In particular, $\operatorname{rank}(S) = r$.

b) Find the singular value decomposition of S in outer product form, in terms of the λ_i and the v_i.

10. **a)** Show that all singular values of an orthogonal matrix are equal to 1.

b) Let A be an $m \times n$ matrix, let Q_1 be an $m \times m$ orthogonal matrix, and let Q_2 be an $n \times n$ orthogonal matrix. Show that A has the *same singular values* as $Q_1 A Q_2$. [**Hint:** Use HW10#11.]

Remark: This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

11. Let A be a matrix of full column rank and let $A = QR$ be the QR decomposition of A.

a) Show that A and R have the same singular values $\sigma_1, \ldots, \sigma_r$ and the same right singular vectors $v_1, \ldots, v_r$.

b) What is the relationship between the left singular vectors of A and R?

12. Let A be a matrix with first singular value σ_1 and first right singular vector v_1. Recall that the *matrix norm* of A is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$, and is denoted $\|A\|$.

a) Show that $\|Ax\|$ is maximized at $x = v_1$ (subject to $\|x\| = 1$), with maximum value σ_1.

b) Suppose now that A is square and λ is an eigenvalue of A. Show that $|\lambda| \leq \sigma_1$. (You may assume λ is real, although it is also true for complex eigenvalues.)

This shows that *the largest singular value is at least as big as the largest eigenvalue.*

13. **a)** Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 \end{pmatrix}.$$

c) Note that A is very close to A' numerically. Were the eigenvalues of A close to the eigenvalues of A'? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are numerically stable*. This is another advantage of the SVD.

14. Decide if each statement is true or false, and explain why.

a) The left singular vectors of A are eigenvectors of A^TA and the right singular vectors are eigenvectors of AA^T.

b) For any matrix A, the matrices AA^T and A^TA have the same nonzero eigenvalues.

c) If S is symmetric, then the nonzero eigenvalues of S are its singular values.

d) If A does not have full column rank, then 0 is a singular value of A.

e) Suppose that A is invertible with singular values $\sigma_1, \ldots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \ldots, \sigma_n + c$.

f) The right singular vectors of A are orthogonal to $\text{Nul}(A)$.