Math 218D-1: Homework #2

due Wednesday, January 25, at 11:59pm

1. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= 2 \\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Use row operations to eliminate x_1 from all but the first equation.

b) Use row operations to modify the system so that x_2 only appears in the first and second equations (and x_1 still only appears in the first).

c) Solve for x_3, then for x_2, then for x_1. What is the solution?

2. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries.

System of Equations	**Matrix Equation**	**Augmented Matrix**
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 + 4x_3 &= 2 \end{aligned}$		
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	
		$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

3. Which of the following matrices are not in row echelon form? Why not?

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

4. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

$$\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$

5. Find values of a and b such that the following system has **a)** zero, **b)** exactly one, and **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - y &= b \end{aligned}$$

6. Give examples of matrices A in *row echelon form* for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?

7. Let A be a matrix in REF. Suppose that A has a pivot in every *row*. Explain why the linear system $Ax = b$ is consistent, no matter what b is.

8. Which of the following matrices are not in reduced row echelon form? Why not?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

9. Describe all possible nonzero 2×2 matrices in RREF.

10. Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use Rabinoff's Reliable Row Reducer.

$$\textbf{a)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \textbf{b)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right) \qquad \textbf{c)} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \qquad \textbf{e)} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array}\right) \qquad \textbf{f)} \left(\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right)$$

11. Determine *how many* solutions each system of equations has. (Do not find the solutions.) [**Hint:** use Problem 10.]

a) $\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = 2 \end{cases}$ **b)** $\begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 9x_3 = 1 \end{cases}$

c) $\begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$

12. Use Gaussian elimination and back-substitution or Jordan substitution to solve

a) $\begin{cases} x_1 + x_2 = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ x_2 + 2x_3 = 3 \end{cases}$ **b)** $\begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 8x_3 = 1. \end{cases}$

What happens if you replace 8 by 9 in (b)?

13. The parabola $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,9)$, $(-1,6)$. Find the coefficients a, b, c.

14. Use the formula for the 2×2 inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

15. Compute the inverses of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why.

a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 4 \\ -3 & 1 & 4 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

d) $\begin{pmatrix} 6 & -4 & -7 & -1 \\ 7 & 0 & 1 & 3 \\ -1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$

16. Consider the linear system

$$\begin{aligned} x_1 + x_2 &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the Problem 15 to solve for x_1, x_2, x_3 in terms of b_1, b_2, b_3.

17. Suppose that

$$A\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\qquad A\begin{pmatrix}-1\\3\\2\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\qquad A\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

What is A^{-1}?

18. Decide if each statement is true or false, and explain why.

a) If A and B are invertible $n \times n$ matrices, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.

b) If A is invertible then so is A^{10}.

c) An $n \times n$ matrix with n pivots is invertible.

d) An invertible $n \times n$ matrix has n pivots.