Math 218D-1: Homework #3

due Wednesday, February 1, at 11:59pm

1. Consider a system of 3 equations in 4 variables. Write the elementary matrices that accomplish the following row operations:

a) $R_2 \mathrel{+}= 2R_1$ **b)** $R_1 \mathrel{-}= \frac{1}{2}R_3$ **c)** $R_3 \times= 2$

d) $R_3 \div= 2$ **e)** $R_1 \longleftrightarrow R_3$ **f)** $R_1 \longleftrightarrow R_2$

2. Consider a system of 3 equations in 4 variables. Write the row operations that the following elementary matrices perform on that system:

a) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ **e)** $\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. For each elementary matrix in Problem 2, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row op}} R_2 \mathrel{+}= R_1 \xrightarrow{\text{undo}} R_2 \mathrel{-}= R_1 \xrightarrow{\text{matrix}} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

a) Explain how to reduce A to a matrix U in REF using three row replacements.

b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i:

$$U = \underline{\hspace{2cm}} A.$$

c) Fill in the blank with a product involving the E_i^{-1}:

$$A = \underline{\hspace{2cm}} U$$

d) Evaluate that product to produce a lower-triangular matrix L with ones on the diagonal such that $A = LU$.

When multiplying elementary matrices, just use row operations!

e) Explain how to reduce U to the 3×3 identity matrix using three more row operations E_4, E_5, E_6.

f) Fill in the blank with a product involving the E_i:

$$A^{-1} = \underline{\hspace{3cm}}.$$

5. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}.$$

a) Perform Gaussian elimination on A without using any row swaps. Write the REF matrix U you obtained.

b) Write the elementary matrices E_1, E_2, E_3 for the row operations you did in (a), with E_1 corresponding to the first row operation.

c) Compute the matrix $L = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$. [**Hint:** Don't multiply matrices! Recall that left-multiplication by E_i^{-1} "un-does" the ith row operation.]

d) Verify that L is lower-unitriangular and that $A = LU$.

6. Solve the following matrix equations by forward- and back-substitution, using the provided LU decomposition. Check your answers by evaluating Ax.

a)
$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

b)
$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

c)
$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

7. Compute the $A = LU$ decomposition of the following matrices using the 2-column method. Check your answers by multiplying LU.

a) $\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$

8. Solve the following matrix equations by forward- and back-substitution, using the provided $PA = LU$ decomposition. Check your answers by evaluating Ax.

a)
$$\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b)
$$\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} x = \begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

9. Compute a $PA = LU$ decomposition for each of the following matrices, using the 3-column method and performing *maximal partial pivoting*. Check your answers by multiplying PA and LU.

a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$

10. Recall that a *permutation matrix* is a product of elementary matrices for row swaps.

a) If P is the $n \times n$ elementary matrix for a row swap, explain why $P^{-1} = P = P^T$.

b) If P is any permutation matrix, show that $P^{-1} = P^T$. [**Hint:** write $P = P_1 P_2 \cdots P_r$ for elementary row swaps P_i.] Is $P = P^T$ for a general permutation matrix?

11. Express each system of linear equations as a vector equation. For example,

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 - x_2 &= 4 \end{aligned} \quad \rightsquigarrow \quad x_1\begin{pmatrix}1\\-1\end{pmatrix} + x_2\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}.$$

a) $\begin{cases} 3x_1 + 2x_2 + 4x_3 = 9 \\ -x_1 + 4x_3 = 2 \end{cases}$ **b)** $\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$

c) $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

12. For each matrix A and vector b, decide if the system $Ax = b$ is consistent. If so, find the parametric form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \rightsquigarrow \quad x_1 = x_2 + 1.$$

Also answer the following questions: Which variables are free? How many solutions does the system have?

a) $A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad b = \begin{pmatrix}1\\1\end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix}3\\2\\49\end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix}3\\2\\48\end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix}2\\4\\-6\\10\end{pmatrix}$

e) $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix}2\\5\\4\end{pmatrix}$

13. For each matrix A and vector b in Problem 12, find the parametric *vector* form of the general solution of $Ax = b$ (if the system is consistent). For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + x_2\begin{pmatrix}1\\1\end{pmatrix}$$

What is the dimension of the solution set?

14. **a)** Is $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$?
If so, what are the weights?

b) Find a vector that is *not* a linear combination of the columns of the matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}.$$

[**Hint:** for both parts, compare Problem 12.]

15. The equation $x + 2y = z$ determines a plane in $\mathbf{R}^3$. (This is an *implicit equation* for the plane).

a) What is the coefficient matrix A for this system?

b) Which are the free variables?

c) Write the parametric form of the solutions of $x + 2y = z$. This expresses the points on the plane in terms of two *parameters*.

d) Do the same for the plane defined by $2y = z$. What is different?

16. The equations

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \end{aligned}$$

determine a line $\mathbf{R}^3$. (These are *implicit equations* for the line). Write the line in parameterized form: that is, find three linear functions $f_1(t), f_2(t), f_3(t)$ in one variable such that all points on the line have the form $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ for a unique value of t. (Use the free variable as the parameter t.)

17. Describe and compare (geometrically) the solution sets of the following systems:

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases} \qquad \begin{cases} 6x_1 + 3x_2 + 2x_3 = 2 \\ 2x_1 + x_2 = 1 \end{cases}$$

This is much easier if you write the solutions in parametric vector form.

18. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix} 1+t \\ 2-t \\ t \end{pmatrix} \qquad t \in \mathbf{R}.$$