Math 218D-1: Homework #4

due Wednesday, February 8, at 11:59pm

1. Decide if each statement is true or false, and explain why.

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An $m \times n$ matrix has at most m pivots.

d) A wide matrix (more columns than rows) must have a free variable.

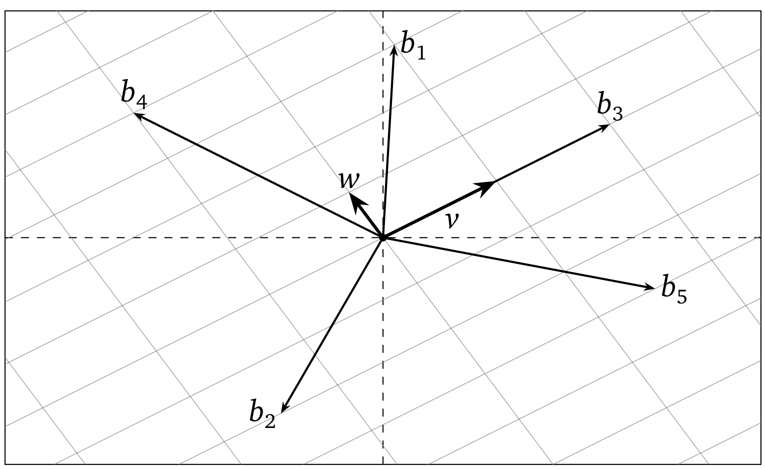
e) If A is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.

2. Consider the vectors

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Draw the 16 linear combinations $cv + dw$ $(c, d = -1, 0, 1, 2)$ as *points* in the xy-plane.

3. Certain vectors v, w in $\mathbf{R}^2$ are drawn below. Express each of b_1, b_2, b_3, b_4, b_5 as a linear combination of v, w. *Do not try to guess the coordinates of v and w!*



4. Consider the vectors

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Draw a picture of all of the linear combinations $au + bv$ for real numbers a, b satisfying $0 \le a \le 1$ and $0 \le b \le 1$. (This will be a shaded region in the xy-plane.)

5. Draw a picture of all vectors $b \in \mathbf{R}^2$ for which the equation

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} x = b$$

is consistent. [**Hint:** the answer is a span!]

6. For each matrix A and vector b, and express the solution set in the form

$$p + \operatorname{Span}\{???\}$$

for some vector p. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

[**Hint:** You found the parametric vector form in HW3#13.]

a) $$A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

c) $$A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

d) $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

7. For each matrix A in Problem 6, write the solution set of $Ax = 0$ as a span. Does there exist a nontrivial solution? Do not do Gauss–Jordan elimination again!

8. When is the following system consistent?

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= b_1 \\ -4x_1 - 5x_2 + 5x_3 &= b_2 \\ 6x_1 + x_2 + 12x_3 &= b_3 \end{aligned}$$

Your answer should be a single linear equation in b_1, b_2, b_3. [**Hint:** perform Gaussian elimination.]

Explain the relationship between this equation

$$\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}\right\}.$$

9. Let A be a 3×4 matrix whose columns span the plane $x + y + z = 0$.

a) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *consistent*.

b) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *inconsistent*.

10. Suppose that $Ax = b$ is consistent. Explain why $Ax = b$ has a unique solution precisely when $Ax = 0$ has only the trivial solution.

11. Give geometric descriptions of the following spans (line, plane, ...).

a) $\text{Span}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **c)** $\text{Span}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\-6\end{pmatrix}\right\}$

d) $\text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ **e)** $\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$

[**Hint:** for **d)**, compare Problem 8.]

12. **a)** List five nonzero vectors contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$.

b) Is $\begin{pmatrix}0\\3\\6\end{pmatrix}$ contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$?

If so, express $\begin{pmatrix}0\\3\\6\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}$.

c) Show that $\begin{pmatrix}7\\8\\9\end{pmatrix}$ is contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$.

d) Describe $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ geometrically.

e) Find a vector not contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$.

13. Decide if each statement is true or false, and explain why.

a) A vector b is a linear combination of the columns of A if and only if $Ax = b$ has a solution.

b) There is a matrix A such that $Ax = \binom{2}{2}$ has infinitely many solutions and $Ax = \binom{2}{-2}$ has exactly one solution.

c) The zero vector is contained in every span.

d) The matrix equation $Ax = 0$ can be consistent or inconsistent, depending on what A is.

e) If the zero vector is a solution of a system of equations, then the system is homogeneous.

f) If $Ax = b$ has a unique solution, then A has a pivot in every column.

g) If $Ax = b$ is consistent, then the solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

h) It is possible for $Ax = b$ to have exactly 13 solutions.

14. Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

$$\textbf{a)}\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

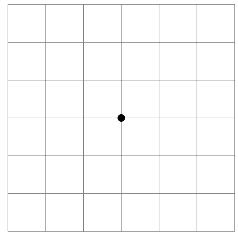
$$\textbf{c)}\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad \textbf{d)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** Compare Problem 7.]

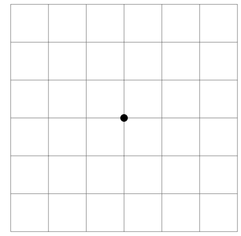
15. Draw pictures of the null space and the column space of the following matrices. Be precise!

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

Nul(A)

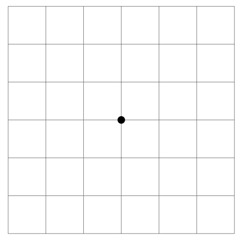


Col(A)

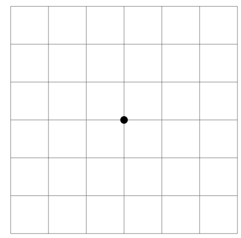


b) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

Nul(A)

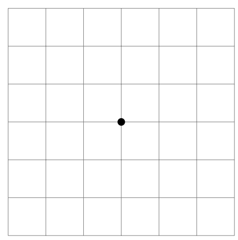


Col(A)

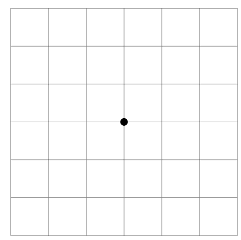


c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

Nul(A)



Col(A)



16. Give examples of subsets V of $\mathbf{R}^2$ such that:

a) V is closed under addition and contains 0, but is not closed under scalar multiplication.

b) V is is closed under scalar multiplication and contains 0, but is not closed under addition.

c) V is closed under addition and scalar multiplication, but does not contain 0.

Therefore, none of these conditions is redundant.

17. Which of the following subsets of $\mathbf{R}^3$ are subspaces? If it is not a subspace, why not? If it is, write it as the column space or null space of some matrix.

a) The plane $\{(x,y,x)\colon x,y\in\mathbf{R}\}$.

b) The plane $\{(x,y,1)\colon x,y\in\mathbf{R}\}$.

c) The set consisting of all vectors (x,y,z) such that $xy=0$.

d) The set consisting of all vectors (x,y,z) such that $x\leq y$.

e) The span of $(1,2,3)$ and $(2,1,-3)$.

f) The solution set of the system of equations $\begin{cases} x+y+z=0 \\ x-2y-z=0. \end{cases}$

g) The solution set of the system of equations $\begin{cases} x+y+z=0 \\ x-2y-z=1. \end{cases}$

18. Find a nonzero 2×2 matrix such that $A^2=0$.

19. **a)** Explain why $\mathrm{Col}(AB)$ is contained in $\mathrm{Col}(A)$.

b) Give an example where $\mathrm{Col}(AB)\neq\mathrm{Col}(A)$.
[**Hint:** use Problem 18.]

20. **a)** Explain why $\mathrm{Nul}(AB)$ contains $\mathrm{Nul}(B)$.

b) Give an example where $\mathrm{Nul}(AB)\neq\mathrm{Nul}(B)$.
[**Hint:** use Problem 18.]

21. Decide if each statement is true or false, and explain why.

a) The column space of an $m\times n$ matrix with m pivots is a subspace of $\mathbf{R}^m$.

b) The null space of an $m\times n$ matrix with n pivots is equal to $\mathbf{R}^n$.

c) If $\mathrm{Col}(A)=\{0\}$, then A is the zero matrix.

d) The column space of $2A$ equals the column space of A.

e) The null space of $A+B$ contains the null space of A.

f) If U is an echelon form of A, then $\mathrm{Nul}(U)=\mathrm{Nul}(A)$.

g) If U is an echelon form of A, then $\mathrm{Col}(U)=\mathrm{Col}(A)$.