Math 218D-1: Homework #6

due Wednesday, February 22, at 11:59pm

1. Consider the following vectors:

$$u = \begin{pmatrix} -.6 \\ .8 \end{pmatrix} \quad v = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a) Compute the lengths $\|u\|, \|v\|$, and $\|w\|$.

b) Compute the lengths $\|2u\|, \|-v\|$, and $\|3w\|$.

c) Find the unit vectors in the directions of u, v, and w.

d) Check the Schwartz inequalities $|u \cdot v| \leq \|u\|\,\|v\|$ and $|v \cdot w| \leq \|v\|\,\|w\|$.

e) Find the angles between u and v and between v and w.

f) Find the distance from v to w.

g) Find unit vectors u', v', w' orthogonal to u, v, w, respectively.

2. If $\|v\| = 5$ and $\|w\| = 3$, what are the smallest and largest possible values of $\|v - w\|$? What are the smallest and largest possible values of $v \cdot w$? Justify your answer using the algebra of dot products.

3. **a)** If $v \cdot w < 0$, what does that say about the angle between v and w?

b) Find three vectors u, v, w in the xy-plane such that $u \cdot v < 0$, $u \cdot w < 0$, and $v \cdot w < 0$.

4. Compute a basis for the orthogonal complement of each of the following spans.

a) $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right\}$ **b)** $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\right\}$ **c)** $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}\right\}$

d) $\operatorname{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ **e)** $\operatorname{Span}\{\} = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

f) $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right\}$

5. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\mathrm{Col}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **b)** $\mathrm{Nul}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **c)** $\mathrm{Row}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$

d) $\mathrm{Nul}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ **e)** $\mathrm{Span}\left\{\begin{pmatrix}3\\1\\3\end{pmatrix},\begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **f)** $\mathrm{Col}\begin{pmatrix}1&2&0\\2&1&0\end{pmatrix}$

[**Hint:** solving **a)–d)** requires only one Gauss-Jordan elimination, and **f)** doesn't require any work.]

6. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\{(x,y,x)\colon x,y\in\mathbf{R}\}$.

b) $\{(x,y,z)\in\mathbf{R}^3\colon x=2y+z\}$.

c) The solution set of the system of equations $\begin{cases}x+y+z=0\\x-2y-z=0.\end{cases}$

d) $\{x\in\mathbf{R}^3\colon Ax=2x\}$, where $A=\begin{pmatrix}0&6&8\\\frac12&0&0\\0&\frac12&0\end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x-2y-z=0$ with the xy-plane.

g) The line $\{(t,-t,t)\colon t\in\mathbf{R}\}$.

[**Hint:** Compare HW5#8.]

7. Construct a matrix A with each of the following properties, or explain why no such matrix exists.

a) The column space contains $(0,2,1)$, and the null space contains $(1,-1,2)$ and $(-1,3,2)$.

b) The row space contains $(0,2,1)$, and the null space contains $(1,-1,2)$ and $(-1,3,2)$.

c) $Ax=\begin{pmatrix}1\\2\\3\end{pmatrix}$ is consistent, and $A^T\begin{pmatrix}-1\\-1\\2\end{pmatrix}=0$.

d) A 2×2 matrix A with no zero entries such that every row of A is orthogonal to every column.

e) The sum of the columns of A is $(0,0,0)$, and the sum of the rows of A is $(1,1,1)$.

8. Suppose that S is a *symmetric matrix*. Explain why $\mathrm{Col}(S) = \mathrm{Nul}(S)^{\perp}$.

9. For each pair of vectors v and w, draw $\mathrm{Span}\{v\}$, and compute and draw the projection p of w onto $\mathrm{Span}\{v\}$.

$$\textbf{a) } v = \begin{pmatrix}1\\0\end{pmatrix},\ w = \begin{pmatrix}\cos(123^\circ)\\ \sin(123^\circ)\end{pmatrix} \qquad \textbf{b) } v = \begin{pmatrix}-1\\1\end{pmatrix},\ w = \begin{pmatrix}1\\1\end{pmatrix}$$

10. For each subspace V and vector b, compute the orthogonal projection b_V of b onto V by solving a normal equation $A^TAx = A^Tb$, and find the distance from b to V.

a) $$V = \mathrm{Col}\begin{pmatrix}1 & 1\\ 1 & 0\\ 0 & 2\end{pmatrix} \qquad b = \begin{pmatrix}1\\4\\3\end{pmatrix}$$

b) $$V = \mathrm{Col}\begin{pmatrix}1 & 2 & 1\\ -1 & 1 & 0\\ 2 & 2 & -1\\ 4 & 3 & 0\end{pmatrix} \qquad b = \begin{pmatrix}-1\\-1\\-1\\7\end{pmatrix}$$

c) $$V = \mathrm{Col}\begin{pmatrix}2 & 2 & -1\\ -4 & -5 & 5\\ 6 & 1 & 12\end{pmatrix} \qquad b = \begin{pmatrix}-6\\-24\\-3\end{pmatrix}$$

11. For each subspace V, compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1, 2, -1)$ with respect to V.

$$\textbf{a) } V = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}\right\} \qquad \textbf{b) } V = \mathrm{Nul}\begin{pmatrix}1 & 2 & 2\\ 0 & 2 & 0\end{pmatrix}$$

$$\textbf{c) } V = \mathbf{R}^3 \qquad \textbf{d) } V = \{0\}$$

[**Hint:** Only part **a)** requires any work.]

12. Compute the orthogonal decomposition $(3, 1, 3) = b_V + b_{V^\perp}$ with respect to each subspace of V of Problem 5(a)–(e).

[**Hint:** Only parts **a)** and **c)** require any work, and even **c)** doesn't require work if you're clever enough. In fact, you can solve all five parts by computing two dot products.]

13. **a)** Let $v, w \in \mathbf{R}^n$. Show that

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

if $v \perp w$.

b) Let V be a subspace of $\mathbf{R}^n$, let $b \in \mathbf{R}^n$, and let $v \in V$. Use **a)** and the fact that $b - b_V \in V^\perp$ to show that

$$\|b - v\|^2 = \|b - b_V\|^2 + \|b_V - v\|^2.$$

Use this to prove that b_V really is the closest vector in V to b.

c) Let V be a subspace of $\mathbf{R}^n$ and let $b \in \mathbf{R}^n$. Use **a)** to show that $\|b_V\| \leq \|b\|$, with equality if and only if $b \in V$.

14. Let A be an $m \times n$ matrix, and let $b \in \mathbf{R}^m$ be a vector. Suppose that $A^T b = 0$. Compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ with respect to $V = \mathrm{Col}(A)$.

15. **a)** Find an implicit equation for the plane

$$\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}.$$

[**Hint:** use Problem 5(a).]

b) Find implicit equations for the line $\{(t, -t, t) : t \in \mathbf{R}\}$.
[**Hint:** use Problem 6(g).]

16. Show that $A^T A = 0$ is only possible when $A = 0$.

17. Let Q be an $n \times n$ matrix such that $Q^T Q = I_n$ (so $Q^T = Q^{-1}$).

a) Show that the columns of Q are unit vectors.

b) Show that the columns of Q are orthogonal to each other.

c) Show that the *rows* of Q are also orthogonal unit vectors.

d) Find all 2×2 matrices Q such that $Q^T Q = I_2$.

Such a matrix Q is called *orthogonal*.[1]

18. Explain why A has full column rank if and only if $A^T A$ is invertible.

[1] I am not responsible for this terminology.

19. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal complements.

b) If A is a 3×4 matrix, then $\mathrm{Col}(A)^{\perp}$ is a subspace of $\mathbf{R}^4$.

c) If A is any matrix, then $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A)$.

d) If A is any matrix, then $\mathrm{Row}(A) = \mathrm{Row}(A^T A)$.

e) If every vector in a subspace V is orthogonal to every vector in another subspace W, then $V = W^{\perp}$.

f) If x is in V and $V^{\perp}$, then $x = 0$.

g) If x is in a subspace V, then the orthogonal projection of x onto V is x.

h) If x is in the orthogonal complement of a subspace V, then the orthogonal projection of x onto V is x.