Math 218D-1: Homework #7

due Wednesday, March 1, at 11:59pm

1. For each column space V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

$$\textbf{a)}\ V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \textbf{b)}\ V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \qquad \textbf{c)}\ V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

2. For each subspace V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

$$\textbf{a)}\ V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\} \qquad \textbf{b)}\ V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

3. For each vector v, compute the projection matrix onto $V = \mathrm{Span}\{v\}$ using the formula $P_V = vv^T/v \cdot v$.

$$\textbf{a)}\ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \textbf{b)}\ v = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} \qquad \textbf{c)}\ v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (\text{in } \mathbf{R}^n)$$

4. For each subspace V, compute the projection matrix P_V.

a) $\{(x, y, x) \colon x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 \colon x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 \colon Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

g) The line $\{(t, -t, t) \colon t \in \mathbf{R}\}$.

[**Hint:** Compare HW5#8 and HW6#6. You can save a lot of work by sometimes computing $P_{V^\perp}$ and using $P_V = I_3 - P_{V^\perp}$.]

5. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 3 \end{pmatrix},$$

and let $V = \mathrm{Col}(A)$.

a) Compute P_V using the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute a basis $\{v_1, v_2\}$ for $V^\perp = \mathrm{Nul}(A^T)$.

c) Let B be the matrix with columns v_1, v_2, and compute $P_{V^\perp}$ using the formula $B(B^TB)^{-1}B^T$.

d) Verify that your answers to (a) and (c) sum to I_4.

(Factor out $ad - bc$ and use a computer to do the matrix multiplication! Your answers should be in fractions, not decimals.)

This illustrates the fact that once you've computed P_V, there's no need to compute $P_{V^\perp}$ separately. It's a lot of extra work!

6. Compute the matrices P_1, P_2 for orthogonal projection onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$, respectively. Now compute P_1P_2, and explain why it is what it is.

7. Consider the plane V defined by the equation $x + 2y - z = 0$. Compute the matrix P_V for orthogonal projection onto V in two ways:

a) Find a basis for V, put your basis vectors into a matrix A, and use the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute the matrix for orthogonal projection $P_{V^\perp}$ onto the line $V^\perp$ using the formula $vv^T/v \cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.

[**Hint:** It doesn't take any work to find a basis for $V^\perp$.]

If V is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

8. Decide if each statement is true or false, and explain why. In each statement, V is a subspace of $\mathbf{R}^n$.

a) The rank of P_V is equal to $\dim(V)$.

b) $P_V P_{V^\perp} = 0$.

c) $P_V + P_{V^\perp} = 0$.

d) $\mathrm{Col}(P_V) = V$.

e) $\mathrm{Nul}(P_V) = V$.

f) $\mathrm{Row}(P_V) = \mathrm{Col}(P_V)$.

g) $\mathrm{Nul}(P_V)^\perp = \mathrm{Col}(P_V)$.

9. Find all least-squares solutions $\widehat{x}$ of each of the following systems of equations $Ax = b$, and compute the projection b_V of b onto $V = \mathrm{Col}(A)$ and the minimum value of $\|A\widehat{x} - b\|$.

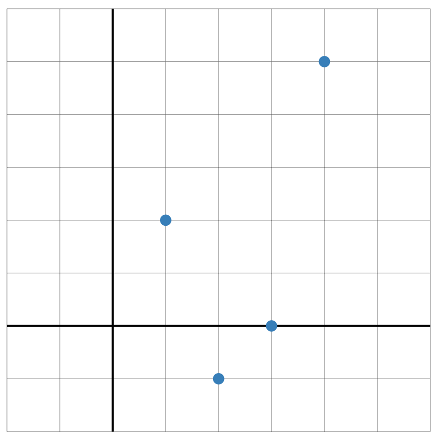
a) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} x = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} x = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$ **d)** $\begin{pmatrix} 3 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix} x = \begin{pmatrix} 9 \\ 7 \\ 7 \end{pmatrix}$

10. Consider the data points

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad p_4 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

a) Find the best-fit line $y = Cx + D$ through these four points, and draw it on the grid below.



b) For each data point $p_i = (a_i, b_i)$, draw the error bar from $(a_i, y(a_i))$ to (a_i, b_i).

c) What is the minimum value of $\sum_{i=1}^{4}(b_i - y(a_i))^2$? How do you know?

d) Verify that the vector

$$\big(2 - y(1),\ -1 - y(2),\ 0 - y(3),\ 5 - y(4)\big)$$

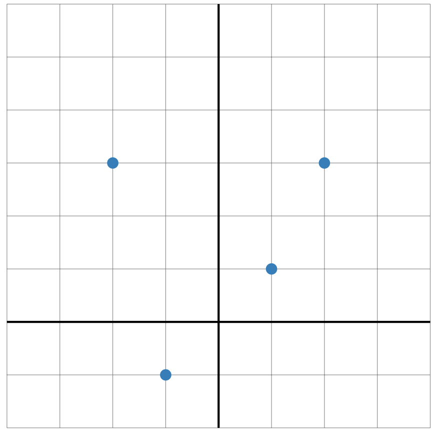
is orthogonal to $(1, 2, 3, 4)$ and $(1, 1, 1, 1)$, and explain why this is necessary.

e) Find the best-fit *horizontal* line $y = D$ through these four points. Verify that D is the average of the y-values of the data points p_1, p_2, p_3, p_4.

11. Consider the data points

$$p_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad p_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad p_4 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

a) Find the best-fit parabola $y = Cx^2 + Dx + E$ through these four points, and draw it on the grid below.



b) For each data point $p_i = (a_i, b_i)$, draw the error bar from $(a_i, y(a_i))$ to (a_i, b_i).

c) What is the minimum value of $\sum_{i=1}^{4}(b_i - y(a_i))^2$? How do you know?

12. Consider the following data points:

$$p_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad p_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \qquad p_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

a) Find the best-fit plane $z = Cx + Dy + E$ through these four points.

b) Interpret the minimized quantity in the situation of this problem.

13. Consider the data points $p_1, \ldots, p_8$:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}, \begin{pmatrix} 2.5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -.5 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2.5 \\ 2 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 3.5 \end{pmatrix}.$$

a) Find the best-fit *ellipse*

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

through these data points.

b) Interpret the minimized quantity in the situation of this problem.

[**Hint:** you can't see it on the graph above, but you can see it on this demo.]

In this problem, I recommend using SymPy (in the Sage cell on the course webpage) or another computer algebra system to do the computations. To solve a normal equation $A^T A x = A^T(1, 2, 3)$, you would use something like

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(A.transpose()*A).solve(A.transpose()*Matrix([1,2,3]))
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Remark: Carl Friedrich Gauss (1777–1865), arguably the greatest mathematician since antiquity, kept food on the table by doing astronomical calculations. He invented much of the linear algebra you are learning in order to compute the trajectories of celestial bodies. Essentially performing the calculations in this problem, he correctly predicted the (elliptical) orbit of the asteroid Ceres as it passed behind the sun in 1801.

14. Suppose that $\widehat{x}$ is a vector such that $A\widehat{x} = (1,1,-1,-1)$. Explain why $\widehat{x}$ is not a least-squares solution of $Ax = (1,1,1,1)$.

15. Decide if each statement is true or false, and explain why.

a) A least-squares solution $\widehat{x}$ of $Ax = b$ is a solution of $A\widehat{x} = b_V$ for $V = \text{Col}(A)$.

b) Any solution of $A^T A\widehat{x} = A^T b$ is a least-squares solution of $Ax = b$.

c) If A has full column rank, then $Ax = b$ has exactly one least-squares solution for every b.

d) If $Ax = b$ has at least one least-squares solution for every b, then A has full row rank.