

The Big Picture

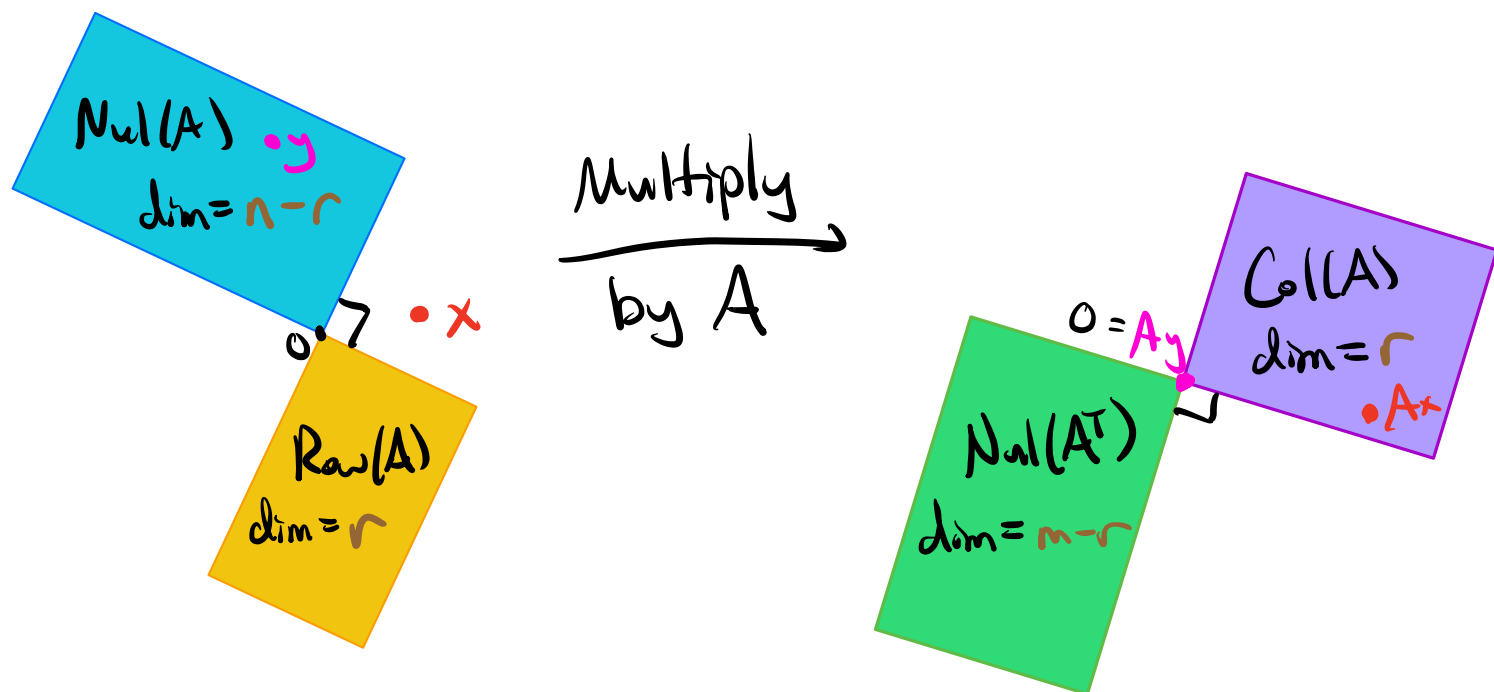
Last time we discussed orthogonality of the 4 subspaces. Here is a summary:

The Big Picture

for an $m \times n$ matrix A of rank r

Row Picture: $\mathbb{R}^n$

Column Picture: $\mathbb{R}^m$



NB: The dimensions match up with $\dim V + \dim V^\perp = n$:

$$\dim \text{Null}(A) + \dim \text{Row}(A) = n$$

$$\dim \text{Null}(A^T) + \dim \text{Col}(A) = m$$



Recall: If A has columns $v_1, \dots, v_n$ then

$$A^T A = \begin{pmatrix} -v_1^T - \\ \vdots \\ -v_n^T - \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \dots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \dots & v_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \dots & v_n \cdot v_n \end{pmatrix}$$

This is the matrix of column dot products:
the (i,j) -entry is $(\text{col } i) \cdot (\text{col } j)$

With orthogonality of the 4 subspaces, we can prove:

Important Fact that we will use many times:

$$\text{Nul}(A^T A) = \text{Nul}(A)$$

Proof: $\text{Nul}(A^T A)$ contains $\text{Nul}(A)$ = (HW 5)

$$x \in \text{Nul}(A) \Rightarrow Ax = 0 \Rightarrow A^T Ax = 0 \Rightarrow x \in \text{Nul}(A^T A)$$

$\text{Nul}(A)$ contains $\text{Nul}(A^T A)$:

$$x \in \text{Nul}(A^T A) \Rightarrow A^T Ax = 0 \Rightarrow Ax \in \text{Nul}(A^T)$$

$$\Rightarrow Ax \in \text{Col}(A) \text{ and } \text{Nul}(A^T)$$

$$\Rightarrow (Ax) \cdot (Ax) = 0 \Rightarrow Ax = 0 \Rightarrow x \in \text{Nul}(A) \quad \checkmark$$

Implicit Equations, Revisited

Recall: $\text{Nul}(A) \xrightarrow{\text{PVF}} \text{Span}\{v_1, \dots, v_{n-r}\}$

takes the **implicit equation** $Ax=0$
and generates the **parametric form**

$$x = a_1 v_1 + \dots + a_{n-r} v_{n-r}. \quad a_1, \dots, a_{n-r} = \text{parameters}$$

Orthogonal complements let us go the other way!

$(\cdot)^\perp$ turns implicit into parametric & vice-versa.

$$\text{Nul}(A)^\perp = \text{Row}(A) \quad \text{Col}(A)^\perp = \text{Nul}(A^T)$$

Recipe: To produce implicit equations for $\text{Col}(A)$:
parametric ↗

(1) Find PVF for $\text{Nul}(A^T)$:

$$\text{Nul}(A^T) \xrightarrow{\text{PVF}} \text{Span}\{v_1, \dots, v_{m-r}\}$$

$$(2) \text{Col}(A) = \text{Nul}(A^T)^\perp$$

$$= \text{Span}\{v_1, \dots, v_{m-r}\}^\perp$$

$$= \text{Nul}\left(\begin{array}{c|c} -v_1^T & \\ \vdots & \\ -v_{m-r}^T & \end{array}\right) \leftarrow \text{implicit}$$

Null Space:
implicit form

Like: easy to check
if $x \in V$: $Ax = 0$

PVF

both require
elimination

$(-)^+$ then PVF
then $(-)^+$

Column Space:
parametric form

Like: can produce
vectors in V :
 $x = a_1 v_1 + \dots + a_n v_n$

Eg: Find an implicit equation for the plane

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{parametric description}$$

$$V^\perp = \text{Nul} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{PVF}} \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\Rightarrow V = \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}^\perp = \text{Nul} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$$
$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \text{implicit equation } -x_1 + x_2 = 0 \right\}$$

Now it's easy to check if a vector is in V :

$-x_1 + x_2 = 0$ means $x_1 = x_2$.

$$\begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} \in V.$$

$$\begin{pmatrix} 3 \\ -3 \\ 7 \end{pmatrix} \notin V.$$

Orthogonal Projections

Recall: to find the best approximate solution of $Ax=b$, want to find the closest vector $\hat{b}$ to b in $\text{Col}(A) = \{Ax : x \in \mathbb{R}^n\}$

Want: $b - \hat{b}$ is orthogonal to $\text{Col}(A)$:

$$b - \hat{b} \in \text{Col}(A)^\perp = \text{Nul}(A^T) \quad [\text{demo}]$$

Def: Let V be a subspace of $\mathbb{R}^n$ and $b \in \mathbb{R}^n$. The orthogonal projection of b onto V is the closest vector b_V in V to b . It is characterized by

$$b - b_V \in V^\perp$$

The orthogonal decomposition of b relative to V is

$$b = b_V + b_{V^\perp}$$

Here $b_{V^\perp} = b - b_V \in V^\perp$. Note that

$$b - b_{V^\perp} = b_V \in V = (V^\perp)^\perp$$

So that $b_{V^\perp}$ is projection onto $V^\perp$.

In other words, the orthogonal decomposition is

$$b = \left(\begin{array}{l} \text{closest vector} \\ \text{to } b \text{ in } V \end{array} b_v \right) + \left(\begin{array}{l} \text{closest vector} \\ \text{to } b \text{ in } V^\perp \end{array} b_{v^\perp} \right)$$
$$b = \left(\begin{array}{l} \text{projection of } b \\ \text{onto } V \end{array} \right) + \left(\begin{array}{l} \text{projection of } b \\ \text{onto } V^\perp \end{array} \right)$$

[demos]

How to compute b_v ?

Step 0: Write V as a column space or a null space.

$V = \text{Col}(A)$: then $V^\perp = \text{Nul}(A^T)$, so

$$b - b_v \in \text{Nul}(A^T) \Rightarrow A^T(b - b_v) = 0$$

If $b_v \in \text{Col}(A)$ then $b_v = A\hat{x}$ for $\hat{x} \in \mathbb{R}^n$:

$$A^T(b - A\hat{x}) = 0 \Rightarrow A^T b - A^T A \hat{x} = 0$$

$$\Rightarrow A^T A \hat{x} = A^T b$$

Solve this equation for $\hat{x} \rightsquigarrow b_v = A\hat{x}$

Eg: Let $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $V = \text{Col} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \leftarrow A$

Find b_V = the orthogonal projection of b to V .

We set up the equations $A^T A \hat{x} = A^T b$:

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \quad \begin{matrix} \text{column} \\ \text{dot} \\ \text{products} \end{matrix}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In augmented matrix form, $A^T A \hat{x} = A^T b$ is:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 2 & 1 \end{array} \right) \xrightarrow{\text{ref}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{array} \right)$$

$$\text{So } \hat{x} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \Rightarrow b_V = A \hat{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\text{Check: } b_{V^\perp} = b - b_V = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \quad [\text{demo}]$$

$$\begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

(columns of A)

$$\Rightarrow b_{V^\perp} \in \text{Col}(A)^\perp$$

$$\text{Distance from } V: \|b - b_V\| = \|b_{V^\perp}\| = \left\| \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \right\| = \frac{1}{\sqrt{2}}$$

$$\text{Orthogonal Decomposition: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

Procedure: To compute the orthogonal projection b_V of b onto $V = \text{Col}(A)$:

(1) Solve the equation $A^T A \hat{x} = A^T b$

(2) $b_V = A \hat{x}$ for **any** solution $\hat{x}$.

Then $b_{V^\perp} = b - b_V$, and the **orthogonal decomposition** of b relative to V is

$$b = b_V + b_{V^\perp}.$$

The **distance** from b to V is $\|b_{V^\perp}\|$.

Eg: Let $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $V = \text{Col} \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & -1 \\ 4 & -1 & -1 \end{pmatrix}$.

Find the orthogonal decomposition of b relative to V .

$$(1) \quad A^T A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \\ -1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & -1 \\ 4 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 6 \\ 6 & 3 & 6 \\ 6 & 6 & 18 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \\ -1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

Solve $A^T A \hat{x} = A^T b$:

$$\left(\begin{array}{ccc|c} 6 & 6 & 6 & 4 \\ 6 & 3 & 6 & -1 \\ 6 & 6 & 18 & 2 \end{array} \right) \xrightarrow{\text{PVE}} \hat{x} = \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

(2) $b_V = A\hat{x}$ for **any** solution. Let's use the particular solution:

$$b_V = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

NB: $b_V = b$: what does that mean?

b was already in V ! More on this later.

Def: The **normal equation** of $Ax=b$ is

$$A^T A \hat{x} = A^T b$$

Fact: $A^T A \hat{x} = A^T b$ is **always consistent**!

(Otherwise the **Procedure** wouldn't work.)

Why? I claim $\text{Col}(A^T) = \text{Col}(A^T A)$.

From before: $\text{Nul}(A) = \text{Nul}(A^T A)$

Take $(-)^{\perp}$: $\text{Nul}(A)^{\perp} = \text{Nul}(A^T A)^{\perp}$

$$\text{Nul}(A)^{\perp} = \text{Row}(A) = \text{Col}(A^T)$$

$$\begin{aligned} \text{Nul}(A^T A)^{\perp} &= \text{Row}(A^T A) = \text{Col}((A^T A)^T) \\ &= \text{Col}(A^T A) \quad \checkmark \end{aligned}$$

Since $A^T b \in \text{Col}(A^T) = \text{Col}(A^T A)$, the equation $A^T A \hat{x} = A^T b$ is consistent. 

NB: If $\hat{x}$ and $\hat{y}$ both solve

$$A^T A \hat{x} = A^T x = A^T A \hat{y}$$

$$\text{then } 0 = A^T A \hat{x} - A^T A \hat{y} = A^T A (\hat{x} - \hat{y})$$

$$\Rightarrow \hat{x} - \hat{y} \in \text{Nul}(A^T A) \stackrel{\text{Fact}}{=} \text{Nul}(A) \Rightarrow A(\hat{x} - \hat{y}) = 0$$

$$\Rightarrow b_v = A \hat{x} = A \hat{y}. \text{ So any soln of } A^T A \hat{x} = A^T b \text{ works.}$$

Now we know how to project onto a column space.

What if $V = \text{Nul}(A)$?

$$\text{Then } V^\perp = \text{Nul}(A)^\perp = \text{Row}(A) = \text{Col}(A^T).$$

So first compute $b_{v^\perp} = \text{projection onto a col space}$,
then $b_v = b - b_{v^\perp}$.

Procedure: To compute the orthogonal projection b_v of b onto $V = \text{Nul}(A)$:

(1) Compute $b_{v^\perp} = \text{projection onto } V^\perp = \text{Col}(A^T)$

$$(2) \quad b_v = b - b_{v^\perp}$$

Use the **symmetry** in the orthogonal decomposition

$$b = b_v + b_{v^\perp}$$

to your advantage!

Eg: Project $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ onto $V = \text{Nul} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

First we project onto $\text{Col} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ ← because which is A & which is A^T

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 2 & 1 \end{array} \right) \xrightarrow{\text{ref}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{array} \right)$$

$$\text{So } \hat{x} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \Rightarrow b_{V^\perp} = A^T \hat{x} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow b_V = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \quad \checkmark$$

Projection onto a Line:

Suppose $V = \text{Span}\{v\}$.

Then $V = \text{Col}(A)$ where $A = v$ (one column).

$$A^T A = v^T v = v \cdot v \quad \text{is a } 1 \times 1 \text{ matrix}$$

$$A^T b = v^T b = v \cdot b$$

so the normal equation becomes

$$A^T A \hat{x} = A^T b \rightsquigarrow (v \cdot v) \hat{x} = v \cdot b$$

$$\text{Then } \hat{x} = \frac{v \cdot b}{v \cdot v} \rightsquigarrow b_V = A \hat{x} = \frac{v \cdot b}{v \cdot v} v$$

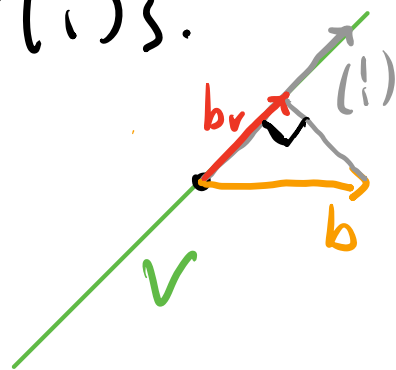
Projection onto the Line $\text{Span}\{v\}$

$$b_v = \frac{v \cdot b}{v \cdot v} v$$

Eg: Project $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ onto $V = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

$$b_v = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

[demo]



Eg: Compute b_v where

$$V = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\} \quad b = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

Note V is a plane in $\mathbb{R}^3 \hookrightarrow V^\perp$ is a line.

In fact, $V = \text{Nul}(1 \ 1 \ 1) \Rightarrow V^\perp = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

Much easier to compute $b_{v^\perp} = \text{proj}$ onto a line.

$$b_{v^\perp} = \frac{b \cdot v}{v \cdot v} v = \frac{\begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{-3}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow b_v = b - b_{v^\perp} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

[demo]

Hint: Ask yourself: is it easier to compute b_v or $b_{v^\perp}$?