

Determinants & Cofactors

Last time: we defined determinants using row ops.

(1) If $A \xrightarrow{R_i + cR_j} B$ then $\det(A) = \det(B)$.

(2) If $A \xrightarrow{R_i \times c} B$ then $\det(A) = \frac{1}{c} \det(B)$.

(3) If $A \xrightarrow{R_i \leftrightarrow R_j} B$ then $\det(A) = -\det(B)$.

(4) $\det(I_n) = 1$.

This is the fastest algorithm for computing the det of a general matrix with known entries. But what if the matrix has unknown entries? This becomes tedious because you don't know if an entry is a pivot!

Eg: $\det \begin{pmatrix} -\lambda & 1 & 3 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = ?$ Is $-\lambda$ a pivot?

Cofactor expansion is a handy recursive formula for the determinant that is useful in this setting.

Recursive: Compute $\det(n \times n)$ by computing several $\det((n-1) \times (n-1))$.

Def: Let A be an $n \times n$ matrix.

- The (i,j) minor A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row & j^{th} column.

- The (i,j) cofactor C_{ij} is

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

- The cofactor matrix is the matrix C whose (i,j) entry is C_{ij} .

Eg: $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $A_{21} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$

$$C_{21} = (-1)^{2+1} \det \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} = -(-3) = 3$$

NB: $(-1)^{i+j}$ follows a checkerboard pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad \begin{array}{l} +: (-1)^{i+j} = 1 \\ -: (-1)^{i+j} = -1 \end{array}$$

Thm (Cofactor Expansion): A is an $n \times n$ matrix, $a_{ij} = (i,j)$ entry of A , $C_{ij} = (i,j)$ cofactor.

(1) Cofactor expansion along the i^{th} row:

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

(2) Cofactor expansion along the j^{th} column:

$$\det(A) = \sum_{i=1}^n a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$$

Eg: $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

- Expand cofactors along the 3rd row:

$$\det(A) = 1 \cdot \det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} + 1 \cdot -\det \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= 1 \cdot (1 - 6) - 1 \cdot (-3) = -2$$

- Expand cofactors along the 2nd column:

$$\det(A) = 1 \cdot -\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} + 1 \cdot -\det \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= 1 \cdot -(-1) + 2 \cdot (-3) + 1 \cdot -(-3) = 1 - 6 + 3 = -2$$

Remarks:

- (1) This is a recursive formula: $C_{ij} = \det((n-1) \times (n-1))$
- (2) You can compute $C_{ij} = (-1)^{i+j} \det(A_{ij})$ however you like: you'll always get the same number
- (3) Expanding along any row or column gives you $\det(A)$ — always the same number.
- (4) This is handy when your matrix has unknown entries or a row/col with a lot of zeros — otherwise it's ridiculously slow = $O(n! \cdot n)$.

Eg: $\det \begin{pmatrix} -\lambda & 1 & 3 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$

expand
1st col

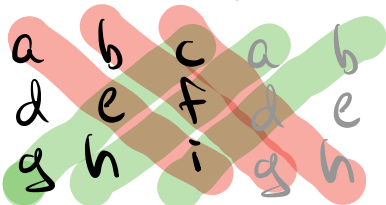
$$\begin{aligned} & (-\lambda) \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & 3 \\ 1 & -\lambda \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & 3 \\ 2-\lambda & 1 \end{pmatrix} \\ &= -\lambda((2-\lambda)(-\lambda) - 1) + 1 \cdot (-\lambda - 3) + 1 \cdot (1 - 3(2-\lambda)) \\ &= -\lambda(-2\lambda + \lambda^2 - 1) + (-\lambda - 3) + 1 - 3(2-\lambda) \\ &= -\lambda^3 + 2\lambda^2 + 5\lambda - 2 \end{aligned}$$

In fact, for 3×3 matrices it's not so hard to compute the determinant when all entries are unknown:

$$\begin{aligned} \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - ge) \\ &= aei + bfg + cdh - afh - bdi - ceg \end{aligned}$$

How to remember this?

Sarrus' Scheme:



To compute $\det(3 \times 3 \text{ matrix})$:
 $\det = aei + bfg + cdh$
 $- ceg - afh - bdi$

Sum the products of forward diagonals, subtract products of backwards diagonals.

Eg: $\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 0 \cdot 2 \cdot 0 + 1 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 1 - 1 \cdot 2 \cdot 3 - 1 \cdot 1 \cdot 0 - 0 \cdot 1 \cdot 1$

$= 4 - 6 = -2$

Warning: This only works for 3×3 matrices!
 → See the big formula at the end for $n \times n$ matrices.

Eg: $\det \begin{pmatrix} 2 & 5 & -3 & -1 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}$ column with lots of zeros

$$= -1 \cdot -\det \begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} - 5 \cdot \det \begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} + 0 \cdot -\det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} + 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix}$$

$$= 1(-24) - 5(11) = -24 - 55 = -79$$

only computed two 3×3 det

Better: Do a row operation first!

$$\det \begin{pmatrix} 2 & 5 & -3 & -1 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow 5R_1} \det \begin{pmatrix} 2 & 5 & -3 & -1 \\ -12 & -28 & 17 & 0 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}$$

$$= -1 \cdot -\det \begin{pmatrix} -12 & -28 & 17 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} = -79$$

only computed one 3×3 det

Methods for Computing Determinants

(1) Special formulas (2×2 , 3×3)

→ best for small matrices, except 3×3 with lots of 0's

(2) Cofactor expansion

→ best if you have unknown entries, or a row/column with lots of zeros.

(3) Row (& column) operations

→ best if you have a big matrix with no unknown entries & no row or column with lots of zeros.

(4) Any combination of the above

→ eg. do a row op. to create a column with lots of zeros, then expand cofactors, ...

Thm: Let C be the cofactor matrix of A . Then

$$AC^T = \det(A) I_n = C^T A$$

In particular, if $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} C^T \quad \text{see supplement}$$

→ Ridiculously inefficient computationally.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

→ generalizes the formula for 2×2 inverse

Cross Products

This is an operation you can do to vectors in $\mathbb{R}^3$.

Recall: the unit vectors in $\mathbb{R}^3$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Def: Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $w = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \in \mathbb{R}^3$.

The cross product is

$$v \times w = \begin{pmatrix} bf - ec \\ cd - af \\ ae - bd \end{pmatrix} \in \mathbb{R}^3$$

So the cross product is $(\text{vector}) \times (\text{vector}) \rightarrow (\text{vector})$

Here's how you remember it:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \text{"det} \begin{pmatrix} \overbrace{e_1 \ e_2 \ e_3}^{\text{expand cofactors}} \\ a \ b \ c \\ d \ e \ f \end{pmatrix} \text{"}$$

$$= e_1 \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} - e_2 \det \begin{pmatrix} a & c \\ d & f \end{pmatrix} + e_3 \det \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

$$= (bf - ec)e_1 - (af - cd)e_2 + (ae - bd)e_3$$

$$= \begin{pmatrix} bf - ec \\ cd - af \\ ae - bd \end{pmatrix}$$

$$\text{Eg: } \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= e_1 \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - e_2 \det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + e_3 \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= -e_1 + e_2 - e_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

Def: Let $u, v, w \in \mathbb{R}^3$. The **triple product** is

$$u \cdot (v \times w) = \det \begin{pmatrix} -u^T \\ -v^T \\ -w^T \end{pmatrix}$$

Check: if $v = (a, b, c)$ $w = (d, e, f)$ $u = (g, h, i)$ then

$$u \cdot (v \times w)$$

$$= \begin{pmatrix} g \\ h \\ i \end{pmatrix} \cdot \left(e_1 \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} - e_2 \det \begin{pmatrix} a & c \\ d & f \end{pmatrix} + e_3 \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} \right)$$

$$= g \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} - h \det \begin{pmatrix} a & c \\ d & f \end{pmatrix} + i \det \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

$$= \det \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \quad \checkmark$$

$$\text{Eg: } u = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v \times w = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \quad u \cdot (v \times w) = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 1 - 3 = -2$$

$$\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = -2$$

Properties:

(1) $v \times w \perp v$ and $v \times w \perp w$

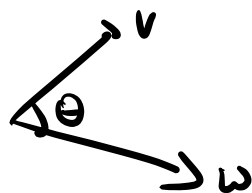
→ because $v \cdot (v \times w) = \det \begin{pmatrix} v^T \\ v^T \\ w^T \end{pmatrix} = 0$

(2) $w \times v = -v \times w$

→ because $\det \begin{pmatrix} e_1 & e_2 & e_3 \\ w^T \\ v^T \end{pmatrix} \xrightarrow{\text{row swap}} -\det \begin{pmatrix} e_1 & e_2 & e_3 \\ v^T \\ w^T \end{pmatrix}$

(3) $\|v \times w\| = \|v\| \cdot \|w\| \cdot \sin(\theta)$

→ compare $v \cdot w = \|v\| \cdot \|w\| \cdot \cos(\theta)$

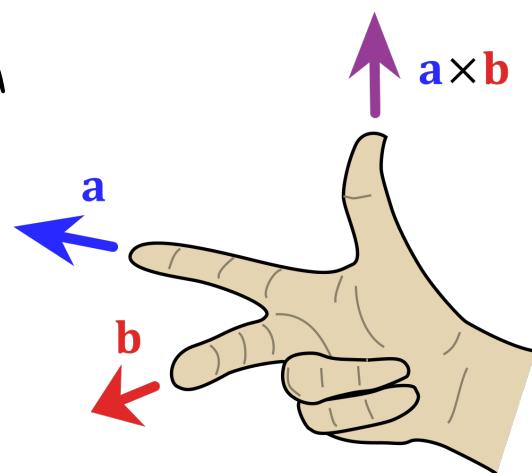


(4) $v \times w = 0 \iff v, w$ are **collinear**

(then $\theta = 0$ or $180^\circ \iff \sin(\theta) = 0$)

(5) $v \times w$ points in the direction determined by the

right hand rule.



NB: (1), (3), & (5) characterize $v \times w$.

The Big Formula

This is an explicit formula for $\det(A)$.

It's useful for some things but not practical — it has $n!$ terms!

Def: A **permutation** of $\{1, \dots, n\}$ is a **re-ordering**

$$\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$$

$\sigma(i)$ = new number in i^{th} position

Eg:

1	2	3	4
↓	↓	↓	↓
3	1	2	4

$$\sigma(1) = 3$$

$$\sigma(3) = 2$$

$$\sigma(2) = 1$$

$$\sigma(4) = 4$$

Q: how many permutations of $\{1, \dots, n\}$ are there?

- n choices for 1st spot
- $(n-1)$ choices for 2nd spot
- $\vdots$
- 1 choice for last spot

$$\text{So } n \cdot (n-1) \cdot \dots \cdot (1) = n!$$

Eg: $n=3$: $123 \rightarrow$

123 132 213 231 312 321

$$6 = 3 \cdot 2 \cdot 1 = 3!$$

Def: A **transposition** is a permutation that just swaps two numbers.

Eg: $123 \rightarrow 132 \quad 213 \quad 321$

Fact: Any permutation can be obtained by doing some number of transpositions.

Def: The **sign** of a permutation σ is $\text{sign}(\sigma) =$

- $+1$ if it can be obtained by doing an **even** number of transpositions.
- -1 if it can be obtained by doing an **odd** number of transpositions.

Eg: $1234 \rightarrow 3124$:

$1234 \rightarrow 3214 \rightarrow 3124$

2 transpositions $\Rightarrow$ sign is **$+1$** .

Thm (Big Formula): Let A be an $n \times n$ matrix with (i,j) entry a_{ij} .

$$\det(A) = \sum_{\sigma} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Eg: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Six permutations: $123 \rightarrow$

123 sign = 1 (0 transpositions)

132 sign = -1 (transposition)

213 sign = -1 (transposition)

231 sign = 1 (2 transpositions)

321 sign = -1 (transposition)

312 sign = 1 (2 transpositions)

$$\det(A)$$

$$a_{11}a_{22}a_{33}$$

$$- a_{11}a_{23}a_{32}$$

$$- a_{12}a_{21}a_{33}$$

$$+ a_{12}a_{23}a_{31}$$

$$- a_{13}a_{22}a_{31}$$

$$+ a_{13}a_{21}a_{32}$$

