Solving Systems of Equations using Elimination

Here's a system of 3 equations in 3 variables:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases}$$

How to solve it?

- **Substitution:** solve 1st equation for x_1, substitute into 2nd & 3rd, continue.
- **Elimination:** "combine" the equations to eliminate variables.

Elimination turns out to scale much better (to more equations & variables), so we'll focus on that.

"replace the 2nd equation with the 2nd minus 2× the 1st"

Eg:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \quad \xrightarrow{R_2 -= 2R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$$\xrightarrow{R_3 -= 3R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -5x_2 - 10x_3 &= -20 \end{aligned}$$

Now we have eliminated x_1 from the 2nd & 3rd eq.s

These now form 2 equations in 2 variables: simpler!

$$x_1 + 2x_2 + 3x_3 = 6 \qquad\qquad x_1 + 2x_2 + 3x_3 = 6$$
$$-7x_2 - 4x_3 = 2 \quad \xrightarrow{R_3 - \frac{5}{7}R_2} \quad -7x_2 - 4x_3 = 2$$
$$-5x_2 - 10x_3 = -20 \qquad\qquad -\frac{50}{7}x_3 = -\frac{150}{7}$$

isolated

We eliminated x_2 from the last equation: now it's one equation in one variable. Easy!

We can now solve via back-substitution:

$$-\frac{50}{7}x_3 = -\frac{150}{7} \Rightarrow x_3 = 3.$$

Substitute into 2nd equation:

$$-7x_2 - 4x_3 = 2 \leadsto -7x_2 - 4\cdot 3 = 2$$

isolated

Now solve for x_2:

$$-7x_2 - 12 = 2 \Rightarrow -7x_2 = 14 \Rightarrow x_2 = -2$$

Substitute both into 1st equation:

$$x_1 + 2x_2 + 3x_3 = 6 \leadsto x_1 + 2\cdot(-2) + 3\cdot 3 = 6$$

isolated

Now solve for x_1:

$$x_1 - 4 + 9 = 6 \Rightarrow x_1 = 1$$

Check:
$$1 + 2(-2) + 3(3) = 6$$
$$2\cdot 1 - 3(-2) + 2(3) = 14$$
$$3\cdot 1 + (-2) - 3 = -2$$

✓

NB: In this case there was one solution — since we could isolate each variable, all values were determined.

Does this always work?

Eg: $4x_2 + 3x_3 = 2$ swap

$x_1 + x_2 - x_3 = 3$

$2x_1 - 3x_2 - 6x_3 = -3$

x_1 is already eliminated from R_1. Fix: swap the 1st 2 eqns.

$$\xrightarrow{R_1 \leftrightarrow R_2} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ 2x_1 - 3x_2 - 6x_3 &= -3 \end{aligned}$$

Now eliminate as before:

$$\xrightarrow{R_3 -= 2R_1} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ -5x_2 - 4x_3 &= -9 \end{aligned}$$

$$\xrightarrow{R_3 += \frac{5}{4}R_2} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ -\tfrac{1}{4}x_3 &= -\tfrac{13}{2} \end{aligned}$$

Solve using back-substitution:

isolated $\quad -\frac{1}{4}x_3 = -\frac{13}{2} \implies x_3 = 26$

Substitute into 2nd equation:

isolated $\quad 4x_2 + 3(26) = 2 \implies x_2 = -19$

Substitute both into 1st equation:

isolated $\quad x_1 - 19 - 26 = 3 \implies x_1 = 48$

NB again there is one solution: each variable was isolated in one equation.

Check:

$$\begin{aligned} 4x_2 + 3x_3 &= 2 \\ x_1 + x_2 - x_3 &= 3 \\ 2x_1 - 3x_2 - 6x_3 &= -3 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} 4(-19) + 3(26) &= 2 \\ 48 - 19 - 26 &= 3 \\ 2(48) - 3(-19) - 6(26) &= -3 \end{aligned}$$

✓

Eg:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= -1 \end{aligned} \quad \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 0 \quad - 3x_2 - 6x_3 &= -4 \\ 0 \quad - 6x_2 - 12x_3 &= -8 \end{aligned}$$

can't isolate x_3!

$$\xrightarrow{R_3 -= 2R_2} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 \qquad 0 &= 0 \end{aligned}$$

Are we done? Yes: choose any value for x_3, then back-substitute to find x_1, x_2:

$$-3x_2 = -4 + 6x_3 \implies x_2 = \frac{4}{3} - 2x_3$$

$$x_1 = 1 - 2x_2 - 3x_3 = 1 - \frac{8}{3} + 4x_3 - 3x_3$$

$$x_1 = -5/3 + x_3$$

Eg: $x_3 = 1 \rightsquigarrow x_1 = -2/3, \; x_2 = -2/3$

Check:

$$\begin{aligned} -2/3 - 4/3 + 3 &= 1 \\ -8/3 - 10/3 + 6 &= 0 \\ -14/3 - 16/3 + 9 &= -1 \end{aligned}$$

✓

In this case there are infinitely many solutions. We'll deal with this in Week 3.

Eg:
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= 0 \end{aligned} \quad \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ -6x_2 - 12x_3 &= -7 \end{aligned}$$

tweak previous example

$$\xrightarrow{R_3 -= 2R_2} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 &= 1 \end{aligned}$$

If our original equations were true, then $0=1$.
Thus our system has no solutions.
(last 2 eqns are parallel planes)

Row Operations are the allowed manipulations we can perform on our equations.

(1)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \quad \xrightarrow{R_2 -= 2R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

row replacement

replace R_2 by $R_2 - 2R_1$

(2)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \quad \xrightarrow{R_1 \leftrightarrow R_2} \quad \begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 14 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

row swap

(change order)

(3) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_1 \times= 2} \begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 12 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$

scalar multiplication
(by nonzero scalar)

Obviously if (x_1, x_2, x_3) is a solution before doing a row operation, then it is true after. Eg. row replacement:

$\begin{aligned} x_1 + 2x_2 + 3x_3 &\to 6 = 6 \\ 2x_1 - 3x_2 + 2x_3 &\to 14 = 14 \end{aligned} \xrightarrow{R_2 -= 2R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &\to 6 = 6 \\ -7x_2 - 4x_3 &\to 2 = 2. \end{aligned}$

Fact: All these operations are reversible: if you have a solution (x_1, x_2, x_3) after doing a row operation, then it's also a solution before.

(system of equations) $\xrightarrow[\text{operations}]{\text{row}}$ ((simpler) system of equations)

SAME SOLUTIONS

This was the whole point: we wanted to solve our (original) system of equations!

Questions: How do you undo (reverse):

- $R_1 += R_2$? $R_1 -= R_2$
- $R_1 \times= 2$? $R_1 \div= 2$
- $R_1 \leftrightarrow R_2$? $R_1 \leftrightarrow R_2$

The variables $x_1, x_2, \ldots$ are just placeholders; only their coefficients matter. Let's extract them into a matrix.

Three Ways to Write System of Linear Equations

(1) As a system of equations:

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$

(2) As a matrix equation $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 14 \end{bmatrix}}_{b}$$

If you expand out the product you get

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

which is what we had before.

The coefficient matrix A comes from the coefficients of the variables:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix} \longleftrightarrow \begin{matrix} 1x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{matrix}$$

The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ contains the unknowns or variables.

NB: A is an $m \times n$ matrix where

m = # equations $\qquad b \in \mathbb{R}^m$ ← size m

n = # variables $\qquad x \in \mathbb{R}^n$ ← size n

(3) As an augmented matrix.

This is a notational convenience: just squash A & b together and separate with a line.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \longleftrightarrow \begin{array}{l} 1x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \end{array}$$

$$\| \qquad [A \mid b]$$

Augmented matrices are good for row operations, which only affect the coefficients (not the variables):

$$\begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \end{array} \xrightarrow{R_2 -= 2R_1} \begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ \quad -7x_2 - 4x_3 = 2 \end{array}$$

$$\equiv$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \xrightarrow{R_2 -= 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \end{array}\right]$$

Eg: Let's solve the system from before using augmented matrices:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \xrightarrow{R_2 -= 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 -= 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$$

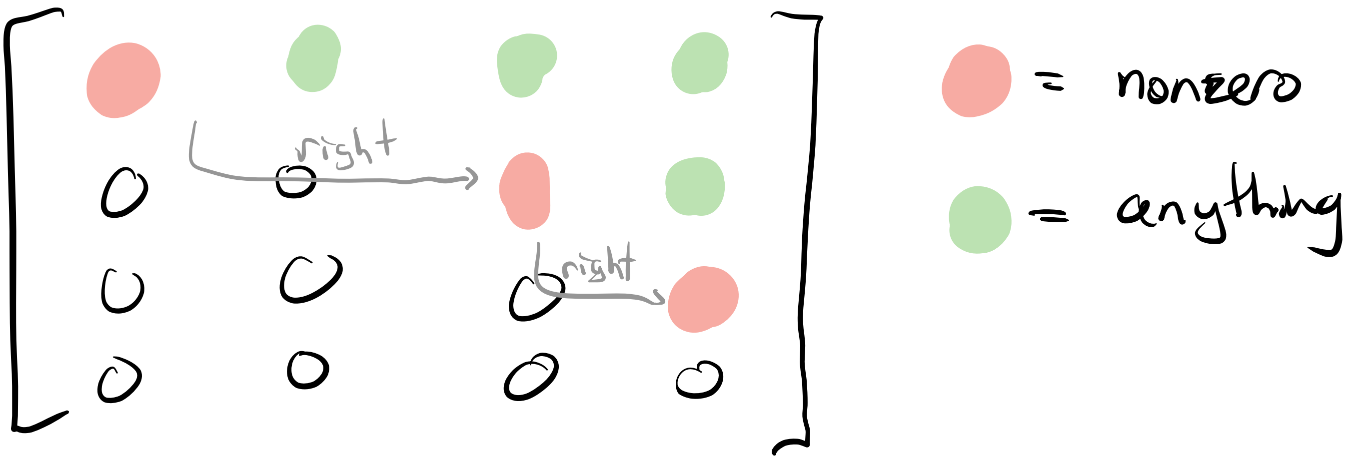
$$\xrightarrow{R_3 -= \frac{5}{7}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

$$\rightsquigarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ -7x_2 - 4x_3 = 2 \\ -\frac{50}{7}x_3 = -\frac{150}{7} \end{cases}$$

Now use back-substitution like before.

What does it mean to be "done"?
(in terms of augmented matrices)

Def: A matrix is in row echelon form (REF) if
(1) The first nonzero entry of each row is to the right of the row above it
(2) All zero rows are at the bottom



REF: $\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$

Not REF: $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 0 & 1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Important: When checking if an augmented matrix is in REF, ignore the augmentation line.

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right]$ REF? $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix}$ ✓

← delete

Think: REF means there's nothing left to eliminate! Each variable is eliminated in later equations, or can't be isolated.

Upshot: The elimination procedure terminates when your (augmented) matrix is in REF.

Solving a linear system $\equiv$ Putting an augmented matrix into REF using row operations

Def: The pivot positions (pivots) of a matrix are the positions of the 1^{st} nonzero entries of each row after you put it into REF.

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

● = pivots

Remarkably, this is well-defined!

Def: The rank of a matrix is the number of pivots it has (in REF).

Eg:

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \xrightarrow[\text{(p.9)}]{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

rank = 3

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{bmatrix} \xrightarrow[\text{(p.4)}]{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2

Number of Solutions (in terms of pivots)

The most basic question you can ask about a system of equations is: how many solutions does it have? This is entirely determined by the pivot positions/pivot columns (columns with a pivot).

(1) The system

pivot columns

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

(p.2)

had one solution. It has a pivot in every column except the augmented column. This means every variable will be isolated when doing back-substitution.

(0) The system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(p.5)

had no solutions. It has a pivot in the augmented column, which leads to the equation $0=1$.

(∞) The system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(p.4)

had infinitely many solutions. It has no pivot in the augmented column and no pivot in the column for the variable x_3. You can't isolate x_3, so you can choose any value.

NB: You have to put the system in REF to find its pivots, so you have to do work to know how many solutions there are.

Def: A system is consistent if it has at least 1 solution (so 1 or ∞). It is inconsistent otherwise.