

Systems of ODEs

Toy Example: Here is an extremely simplistic model of disease spread:

$H(t)$ = # healthy people at time t (in years)

$I(t)$ = # infected people at time t

$D(t)$ = # dead people at time t

Assumptions:

(1) Healthy people are infected at a rate of $0.3 \times \text{\#healthy people}$

(2) Infected people recover at a rate of $0.9 \times \text{\#infected people}$

(3) Infected people die at a rate of $0.1 \times \text{\#infected people}$

In equations:

$$(1) \quad \frac{dH}{dt} = \overset{\text{infected}}{-0.3H} + \overset{\text{recovered}}{0.9I}$$

$$(2) \quad \frac{dI}{dt} = \overset{\text{infected}}{0.3H} - \overset{\text{recovered}}{0.9I} - \overset{\text{dead}}{0.1I}$$

$$(3) \quad \frac{dD}{dt} = \overset{\text{dead}}{0.1I}$$

Matrix Form: let $u(t) = (H(t), I(t), D(t))$.

$$\frac{du(t)}{dt} = u'(t) = \begin{bmatrix} -0.3 & 0.9 & 0 \\ 0.3 & -0.9-0.1 & 0 \\ 0 & 0.1 & 0 \end{bmatrix} u(t)$$

Def: A system of linear ordinary differential equations (ODEs) is a system of equations in unknown functions $u_1(t), \dots, u_n(t)$ equating the derivatives u_i' with a linear combination of the u_i :

$$u_1'(t) = a_{11}u_1(t) + \dots + a_{1n}u_n(t)$$

$\vdots$

$$u_n'(t) = a_{n1}u_1(t) + \dots + a_{nn}u_n(t)$$

Matrix form: writing $u(t) = (u_1(t), \dots, u_n(t))$ and $u'(t) = (u_1'(t), \dots, u_n'(t))$, a system of linear ODEs has the form

$$u'(t) = Au(t)$$

for an $n \times n$ matrix A

(with numbers in it, not functions of t).

If you also specify the initial value $u(0) = u_0$, this is called an initial value problem.

$\uparrow$
some vector

How to solve a system of linear ODEs?

Diagonalize A !

Eg: Suppose u_0 is an eigenvector of A : $Au_0 = \lambda u_0$.
Then the solution of the initial value problem

$$u' = Au \quad u(0) = u_0 \quad \text{is } u(t) = e^{\lambda t} u_0:$$

$$u'(t) = \frac{d}{dt} e^{\lambda t} \overset{\text{does not depend on } t}{u_0} = \lambda e^{\lambda t} u_0$$

$$Au(t) = A \overset{\text{scalar}}{e^{\lambda t}} u_0 = e^{\lambda t} Au_0 = \lambda e^{\lambda t} u_0$$

$$u(0) = e^{\lambda \cdot 0} u_0 = u_0 \quad \checkmark$$

In general, we expand u_0 in an eigenbasis, as for difference equations:

$$u_0 = x_1 \omega_1 + \dots + x_n \omega_n \quad A\omega_i = \lambda_i \omega_i$$

$$\hookrightarrow u(t) = e^{\lambda_1 t} x_1 \omega_1 + \dots + e^{\lambda_n t} x_n \omega_n$$

is the solution of $u' = Au$, $u(0) = u_0$.

Check:

$$u'(t) = \lambda_1 e^{\lambda_1 t} x_1 \omega_1 + \dots + \lambda_n e^{\lambda_n t} x_n \omega_n$$

$$Au(t) = e^{\lambda_1 t} x_1 A\omega_1 + \dots + e^{\lambda_n t} x_n A\omega_n$$

$$= \lambda_1 e^{\lambda_1 t} x_1 \omega_1 + \dots + \lambda_n e^{\lambda_n t} x_n \omega_n$$

$$u(0) = e^{\lambda \cdot 0} x_1 \omega_1 + \dots + e^{\lambda \cdot 0} x_n \omega_n = u_0 \quad \checkmark$$

Eg: In our infectious disease model, suppose
 $u_0 = (1000, 1, 0)$ (1000 healthy people,
1 infected, 0 dead)

Eigenvalues of $A = \begin{pmatrix} -0.3 & .9 & 0 \\ 0.3 & -.1 & 0 \\ 0 & .1 & 0 \end{pmatrix}$ are

$$\lambda_1 \approx -.0235$$

$$\lambda_2 \approx -1.28$$

$$\lambda_3 = 0$$

Eigenvectors are

$$w_1 \approx \begin{pmatrix} 11.77 \\ -12.77 \\ 1 \end{pmatrix} \quad w_2 \approx \begin{pmatrix} -.765 \\ -.235 \\ 1 \end{pmatrix} \quad w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Solve $u_0 = x_1 w_1 + x_2 w_2 + x_3 w_3$:

$$u_0 = \begin{pmatrix} 1000 \\ 1 \\ 0 \end{pmatrix} \approx 18.70 w_1 - 1019.70 w_2 + 1001 w_3$$

Solution is:

$$u(t) = e^{-.0235t} \cdot 18.70 w_1 - e^{-1.28t} \cdot 1019.70 w_2 + 1001 w_3$$

$$H(t) = 220 e^{-.0235t} + 780 e^{-1.28t}$$

$$\Rightarrow I(t) = -238 e^{-.0235t} + 239 e^{-1.28t}$$

$$D(t) = 18.7 e^{-.0235t} - 1019.7 e^{-1.28t} + 1001$$

Looks like the human race is doomed...

Procedure for solving a linear system of ODEs using diagonalization:

To solve $u' = Au$, $u(0) = u_0$ when A is diagonalizable:

- (1) **Diagonalize A :** get an eigenbasis $\{w_1, \dots, w_n\}$ with eigenvalues $\lambda_1, \dots, \lambda_n$.
- (2) **Expand u_0 in the eigenbasis:**
solve $u_0 = x_1 w_1 + \dots + x_n w_n$

Solution:

$$u(t) = e^{\lambda_1 t} x_1 w_1 + \dots + e^{\lambda_n t} x_n w_n$$

Compare to:

Procedure for solving a Difference Equation using diagonalization:

To solve $v_{k+1} = Av_k$, v_0 fixed when A is diagonalizable:

- (1) **Diagonalize A :** get an eigenbasis $\{w_1, \dots, w_n\}$ with eigenvalues $\lambda_1, \dots, \lambda_n$.
- (2) **Expand v_0 in the eigenbasis:**
solve $v_0 = x_1 w_1 + \dots + x_n w_n$

Solution:

$$v_k = \lambda_1^k x_1 w_1 + \dots + \lambda_n^k x_n w_n$$

This works fine with **complex eigenvalues**. As with difference equations, you can write the solution with **real numbers** using trig functions.

Eg: $u_1'(t) = u_2, \quad u_2'(t) = -4u_1,$
 $u_1(0) = 2 \quad u_2(0) = 0$

$$\hookrightarrow u' = Au \quad \text{for} \quad A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \quad u(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Eigenvalues are $\lambda = 2i, \quad \bar{\lambda} = -2i$

Eigenvectors are $w = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \quad \bar{w} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$

Solve $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = x_1 w + x_2 \bar{w} \hookrightarrow x_1 = x_2 = 1$

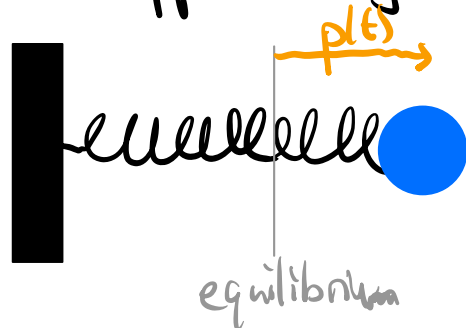
Solution is $(x_1 = x_2 = 1)$

$$\begin{aligned} u(t) &= e^{\lambda t} w + e^{\bar{\lambda} t} \bar{w} = 2\operatorname{Re}[e^{\lambda t} w] \\ &= 2\operatorname{Re}\left[e^{2it} \begin{pmatrix} 1 \\ 2i \end{pmatrix}\right] = 2\operatorname{Re}\left[(\cos(2t) + i\sin(2t)) \begin{pmatrix} 1 \\ 2i \end{pmatrix}\right] \\ &= 2\operatorname{Re}\begin{pmatrix} \cos(2t) + i\sin(2t) \\ -2\sin(2t) + 2i\cos(2t) \end{pmatrix} = \begin{pmatrix} 2\cos(2t) \\ -4\sin(2t) \end{pmatrix} \end{aligned}$$

Check: $u_1' = (2\cos(2t))' = -4\sin(2t) = u_2$
 $u_2' = (-4\sin(2t))' = -8\cos(2t) = -4u_1$
 $u_1(0) = 2 \quad u_2(0) = 0$ ✓

This method can also be used to solve (linear) ODEs containing higher-order derivatives.

Eg: Hooke's Law says the force applied by a spring is proportional to the amount it is stretched or compressed:



$$F(t) = -k p(t) \quad k > 0$$

$F = ma$, $a = \text{acceleration} = p''$: replace k by k/m :

$$p''(t) = -k p(t)$$

Trick: Let $u_1 = p$, $u_2 = p'$. Then

$$u_1' = u_2 \quad u_2' = -k u_1$$

This is the system

$$u'(t) = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} u(t)$$

We solved this before for $k=4$, $u(0) = (2, 0)$:

$$p(t) = 2 \cos(2t)$$

$$p'(t) = -4 \sin(2t)$$

oscillation.

The Matrix Exponential

There are 2 features missing from the ODEs picture that we had for difference equations:

(1) **Matrix form**: $V_k = C D^k C^{-1} V_0$

(2) **Existence of solutions**:

it's obvious that $V_k = A^k V_0$ has a solution
— it was not obvious how to compute it.

Both can be filled in using the matrix exponential.

Recall: Using Taylor expansions, you can write

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \quad (\text{convergent sum})$$

Def: Let A be an $n \times n$ matrix. The **matrix exponential** is the $n \times n$ matrix

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \quad (\text{convergent sum})$$

Eg: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow A^2 = 0$, so

$$e^{At} = I_2 + At + 0 + \dots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Eg: $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \rightsquigarrow A^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$, so

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{pmatrix} + \left(\frac{1}{2!} \lambda_1^2 t^2 \begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \right) + \left(\frac{1}{3!} \lambda_1^3 t^3 \begin{pmatrix} 0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \right) + \dots$$
$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Why do we care about e^{At} ?

Fact: $\frac{d}{dt} e^{At} = A e^{At}$

Consequence: $u(t) = e^{At} u_0$ solves the linear ODE

$$u'(t) = A u(t) \quad u(0) = u_0$$

In particular, a solution exists.

The equations

$$u(t) = e^{At} u_0 \quad \text{and} \quad v_k = A^k v_0$$

are analogous: they both show a solution exists, but give you no way to compute it.

Eg: If $A = CDC^{-1}$ is diagonalizable then

$$e^{At} = C e^{Dt} C^{-1} = C \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} C^{-1}$$

This is computable!

The equations

$$e^{At} = C e^{Dt} C^{-1} \quad \text{and} \quad A^k = C D^k C^{-1}$$

are also analogous: they are computable!

In fact, if you expand out

$$u(t) = C e^{Dt} C^{-1} u_0$$

you exactly get the vector form

$$u(t) = e^{\lambda_1 t} x_1 w_1 + \dots + e^{\lambda_n t} x_n w_n$$

where $(x_1, \dots, x_n) = C^{-1} u_0$.

Difference Equation	Dictionary	Initial Value Problem
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$V_{k+1} = A V_k$	v_0 fixed	Problem
$V_k = A^k V_0$	Uncomputable Solution	$u'(t) = A u(t)$
$V_k = \lambda_1^k x_1 w_1 + \dots + \lambda_n^k x_n w_n$	Computable Solution	$u(t) = e^{At} u(0)$
for $v_0 = x_1 w_1 + \dots + x_n w_n$	(when diagonalizable)	for $u(0) = x_1 w_1 + \dots + x_n w_n$
$A^k = C D^k C^{-1}$	Matrix Form	$e^{At} = C e^{Dt} C^{-1}$
