

# Elementary Matrices

These give a way to do row operations by matrix multiplication!

**Def:** An **elementary matrix** is a matrix obtained from  $I_n$  by doing **one** row operation.

**Eg:**

- $\bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + 2R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \times 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Fact:** If  $E$  is an  $m \times m$  elementary matrix and  $A$  is an  $m \times n$  matrix, then

$$E \cdot A = (\text{what you get by performing that row operation on } A.)$$

$(\text{row operations}) \equiv (\text{left-multiplication by elementary matrices})$

Eg:  $\begin{bmatrix} -2 & -3 & 0 & -7 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{same!}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 0 & -7 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Left-multiplication by  $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  does  $R_1 \leftrightarrow R_2$ .

**Fact:** An elementary matrix is invertible. Its inverse corresponds to the elementary matrix that un-does the row operation.

Why?  $E_1 = (\text{matrix for } R_1 \leftrightarrow R_2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$E_2 = (\text{matrix for } R_1 \leftrightarrow R_2) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 E_2 = E_1 E_2 I_n = E_1 (\text{do } R_1 \leftrightarrow R_2 \text{ to } I_n)$$

$$= \left( \begin{array}{l} \text{first do } R_1 \leftrightarrow R_2 \text{ to } I_n, \\ \text{then do } R_1 \leftrightarrow R_2 \text{ to the result} \end{array} \right) = I_n.$$

Check:  $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

What if you do multiple row operations?

Consider these row operations & their elementary matrices:

$$E_1: R_1 \leftarrow R_1 + 2R_2 \quad E_2: R_2 \leftarrow 2R_2 \quad E_3: R_2 \leftrightarrow R_3$$

Apply in order to  $A$

$$A \xrightarrow{R_1 \leftarrow R_1 + 2R_2} E_1 A \xrightarrow{\substack{R_2 \leftarrow 2R_2 \\ \text{do } R_2 \leftarrow 2R_2 \\ \text{to } E_1 A}} E_2(E_1 A) \xrightarrow{R_2 \leftrightarrow R_3} E_3(E_2(E_1 A)) \\ = (E_3 E_2 E_1) A$$

The elementary matrices ended up in the **opposite order!** Why?

$$E_3 E_2 E_1 A = E_3 E_2 (E_1 A): \text{ first multiply by } E_1 \\ \text{ then by } E_2, \text{ then } E_3$$

## Application to Invertibility:

Suppose  $\text{RREF}(A) = I_n$ .

So there are some number of row ops to transform  $A \rightsquigarrow I_n$ .

Let  $E_1, \dots, E_r$  be their elementary matrices.

$$I_n = (E_r E_{r-1} \dots E_1) A$$

$$\Rightarrow A^{-1} = E_r E_{r-1} \dots E_1$$

In particular,  $A$  is **invertible**: justifies (part of) the **thm** last time.

This also justifies the algorithm for computing  $A^{-1}$ :

$$[A \mid I_n] \xrightarrow{\text{row ops}} [I_n \mid B]$$

$$\text{Then } [I_n \mid B] = (E_r \dots E_1) [A \mid I_n]$$

$$= [(E_r \dots E_1) A \mid (E_r \dots E_1)]$$

column  $\sim$  1st  
matrix  
multiplication

$$\Rightarrow B = E_r \dots E_1 = A^{-1}$$

$$\Downarrow \\ C[A|B] = [C|CB]$$

# Triangular Matrices

Will lead to LU decomposition — computationally efficient way to solve  $Ax=b$  for many values of  $b$ .

Def: A matrix is **upper/lower triangular** if all entries below/above the diagonal are zero.

upper triangular

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \end{bmatrix}$$

$\bullet = \text{anything}$

A matrix is **unitriangular** if it is triangular and all diagonal entries = 1.

upper unitriangular

$$\begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \end{bmatrix}$$

lower unitriangular

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \bullet & 1 & 0 & 0 \\ \bullet & \bullet & 1 & 0 \end{bmatrix}$$

NB: A matrix is **diagonal**  $\Leftrightarrow$  it is both upper- and lower-triangular.

Eg: A matrix in REF is upper- $\Delta$ ular

$$\begin{pmatrix} 1 & 7 & 2 & 4 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Eg:** If  $E$  is the elementary matrix for  $R_i \leftarrow R_i + c \cdot R_j$  for  $i > j$  (add a higher row to a lower row) then  $E$  is lower-unitriangular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E$$

**Fact:** If  $A$  &  $B$  are  $n \times n$  upper (uni)triangular matrices then so are  $AB$  and  $A^{-1}$  (if  $A$  is invertible).

Likewise for lower (uni)triangular.

**Eg:**

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 29 & 11 & 1 \end{pmatrix}$$

**NB:** Unitriangular & square  $\Rightarrow$  invertible

# LU Decompositions

**Fact:** If Gaussian elimination on  $A$  requires no row swaps, then

$$A = L U$$

for  $L$  lower-unitriangular and  $U$  in REF.

**Eg:**  $\begin{pmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 6 & 1 & 0 \end{pmatrix} = A = L U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

Why do we care?

**Recall:** Solving  $Ax=b$  (for  $A$   $n \times n$ ) requires  $\approx \frac{2}{3}n^3$  flops

**Algorithm (solving  $Ax=b$  using  $A=LU$ )**

**Input:** An LU factorization  $A=LU$  & vector  $b$

**Output:** A solution of  $Ax=b$

**Procedure:**

(1) Solve  $Ly=b$  using forward-substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ \bullet & 1 & 0 \\ \bullet & \bullet & 1 \end{bmatrix} y = b \equiv \begin{aligned} y_1 &= b_1 \\ y_2 + \bullet y_1 &= b_2 \\ y_3 + \bullet y_2 + \bullet y_1 &= b_3 \end{aligned}$$

( $n^2 - n$  flops)

(2) Solve  $Ux=y$  using backward-substitution  
( $n^2$  flops)

Then  $Ax = LUx = L(Ux) = Ly = b$  ✓

NB: total complexity is  $\approx 2n^2$  flops  
→ way faster than  $\frac{2}{3}n^3$ !

Eg: Solve  $Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  using

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 6 & 1 & 0 \end{bmatrix} = A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$$

$$(1) Ly = b: \begin{array}{lcl} y_1 & = & 1 \\ 2y_1 + y_2 & = & 0 \\ 3y_1 - y_2 + y_3 & = & 1 \end{array} \quad \begin{array}{l} \text{forward} \\ \text{subst} \end{array} \quad y = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$$

$$(1) Ux = y: \begin{array}{lcl} 2x_1 + x_2 & = & 1 \\ 2x_2 + 4x_3 & = & -2 \\ 4x_3 & = & -4 \end{array} \quad \begin{array}{l} \text{backward} \\ \text{subst} \end{array} \quad x = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Check:  $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 6 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  ✓

Where does  $A = LU$  come from? How to compute it?

If you can run Gaussian elimination with **no row swaps** then  $A \xrightarrow[\text{ops}]{\text{row ops}} U = REF$  using row ops of the form

$$R_i \leftarrow cR_j \quad (i > j) \iff E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \quad R_3 \leftarrow cR_1$$

In this case  $U = (E_r \cdots E_1)A$ ,  $U = REF$

$E_i$  = elementary matrix for  $R_i \leftarrow cR_j$ ,  $i > j$   
(clear **down**)

$E_i$  is **lower-unitriangular**

$\Rightarrow E_r \cdots E_1$  is **lower-unitriangular**

$\Rightarrow L = (E_r \cdots E_1)^{-1}$  is **lower-unitriangular**

and  $LU = \underbrace{(E_r \cdots E_1)^{-1}}_L \underbrace{(E_r \cdots E_1)A}_U = I_n A = A$  ✓

**NB:**  $L = (E_r \cdots E_1)^{-1}$  "records the row operations"  
 $\rightarrow$  keeps track of elimination

**NB:**  $A = LU$  is a **matrix factorization**: it is a way to write a matrix as a product of **simpler** matrices. We will learn many more of these.

Eg:  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 \leftarrow 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 7R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

U

$$L = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

To compute  $E_1^{-1} E_2^{-1} E_3^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} I_3$ ,

start with  $I_3$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$E_3^{-1} = \text{undo } E_3$

$$\xrightarrow{R_3 \leftarrow 7R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$$

$E_2^{-1} = \text{undo } E_2$

$$\xrightarrow{R_2 \leftarrow 4R_1} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$$

$E_1^{-1} = \text{undo } E_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} = L$$

Check:  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

Here's a better way to do the bookkeeping.

Algorithm (LU Decomposition; 2 column method):

Input: A matrix where Gaussian elimination requires no row swaps

Output: A factorization  $A=LU$  for  $L$  lower-unitriangular and  $U$  in REF (the output of Gaussian elimination).

Procedure: Do Gaussian elimination, keeping track of the row operations as follows: start with a blank  $n \times n$  matrix  $L$ .

- for each row replacement  $R_i \leftarrow R_i + c R_j$ , put  $-c$  in the  $(i,j)$  entry of  $L$ .

Add 1's & 0's to the remaining blank entries of  $L$  (1's on the diagonal, 0's elsewhere).

Then

$$A = LU$$

Eg:  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$  ↙ 2 columns ↘

|                       | L                                                                                                                                         | U                                                                                                                                                  |
|-----------------------|-------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------|
|                       | $\begin{bmatrix} \text{blank} \end{bmatrix}$                                                                                              | $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ <span style="color: grey; font-size: 0.8em;">← start with A in U column</span> |
| $R_2 \leftarrow 4R_1$ | $\begin{bmatrix} 4 \end{bmatrix}$                                                                                                         | $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$                                                                              |
| $R_3 \leftarrow 7R_1$ | $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$                                                                                                    | $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$                                                                           |
| $R_3 \leftarrow 2R_2$ | $\begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$                                                                                               | $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$                                                                              |
|                       | $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$ <span style="color: grey; font-size: 0.8em;">↖ add these ↗</span> | U                                                                                                                                                  |

**NB:** Finding  $A=LU$  is just Gaussian elimination + extra bookkeeping  $\rightarrow$  same complexity.  $\approx \frac{2}{3}n^3$  flops  
 Then after computing  $A=LU$ , solving  $Ax=b$  requires  $\approx 2n^2$  flops.

Still have to do elimination once, but then solving  $Ax=b$  for new values of  $b$  is much faster.

Eg: If  $A$  is  $1000 \times 1000$  and we want to solve  $Ax=b$  for 1000 values of  $b$ :

- $\frac{2}{3}$  gigaflops to compute  $A=LU$
- $2 \text{ megaflops} \times 1000 = 2 \text{ gigaflops}$  to solve  $Ax=b$  1000 times.

That's **250x faster** than  $\frac{2}{3}$  teraflops from doing elimination 1000 times!

[demo]

What about inverses?

Wouldn't it be better to compute  $A^{-1}$  and solve  $A^{-1}x=b$  1000 times?

No, for 2 reasons:

(1) Computing  $A^{-1}$  takes  $\approx \frac{4}{3}n^3$  flops = twice as long as the elimination step!

(2) Computing  $A^{-1}$  is not numerically stable (less accurate due to rounding errors).

```

from sympy import *
from time import time
# This is the 10x10 matrix with 2's on the diagonal and 1's elsewhere
# eye(n) = nxn identity matrix; ones(n) = nxn matrix of 1's
# (multiply by 1.0 to force it to use floating point arithmetic)
A = (eye(10) + ones(10)) * 1.0
# This is the vector [1,1,1,1,1,1,1,1,1,1]
b = ones(10, 1) * 1.0
start = time()
# Compute LU decomposition
L, U, _ = A.LUdecomposition()
# Solve using forward- and reverse-substitution 1000 times
for _ in range(1000): U.upper_triangular_solve(L.lower_triangular_solve(b))
end = time()
print(end - start)
# "7.144780397415161" (seconds)

start = time()
# Solve using elimination 1000 times
for _ in range(1000): A.solve(b)
end = time()
print(end - start)
# "48.048372983932495" (seconds)
# Roughly 10x slower!

```

# Maximal Partial Pivoting

Eg:  $\begin{cases} x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$  has one solution:  $x_1 = x_2 = 1$

Let's tweak it a little bit:

$\begin{cases} 10^{-17}x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \rightarrow$  presumably has one solution  $(x_1, x_2) \approx (1, 1)$ .

$$\left[ \begin{array}{cc|c} 10^{-17} & 1 & 1 \\ 1 & 1 & 2 \end{array} \right] \xrightarrow{R_2 \leftarrow 10^{17}R_1} \left[ \begin{array}{cc|c} 10^{-17} & 1 & 1 \\ 0 & 1-10^{17} & 2-10^{17} \end{array} \right]$$

$10^{-17}x_1 + x_2 = 1$

$(1-10^{17})x_2 = 2-10^{17}$

$$\Rightarrow x_2 = \frac{2-10^{17}}{1-10^{17}} = \frac{1+(1-10^{17})}{1-10^{17}} = 1 + \frac{1}{1-10^{17}} \approx 1$$

$$x_1 = 10^{17}(1-x_2) = \frac{10^{17}}{10^{17}-1} \approx 1$$

Let's try this on a computer. [demo]

What went wrong?

→ Most programming languages use 64-digit floating point numbers. That means it has  $\approx 16$  digits of precision.

The computer thinks  $1-10^{17} = -10^{17} = 2-10^{17}$

```

from sympy import *
# 1e-17 is 10-17
A = Matrix([[1e-17, 1.0, 1.0], [1.0, 1.0, 2.0]])
# This does R2 -= 1017 R1
# (force sympy to use the smaller pivot)
A.row_op(1, lambda v, j: v - 1e17*A[0,j])
pprint(A)
#   [1e-17  1  1]
#   [0    -1e17 -1e17]
# Now do Jordan substitution
pprint(A.rref(pivots=False))
#   [1 0 0]
#   [0 1 1]
# This answer is numerically very wrong!

```

$$\begin{bmatrix} 10^{-17} & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftarrow 10^{17} R_1} \begin{bmatrix} 10^{-17} & 1 & 1 & 2 \\ 0 & -10^{17} & -10^{17} & -10^{17} \end{bmatrix}$$

$$\begin{aligned} 10^{-17}x_1 + x_2 &= 1 \\ x_2 &= 1 \end{aligned}$$

$$\leadsto x_1 = 10^{17}(0) = 0$$

$(0, 1) \neq (1, 1)$  : not numerically stable.

What went wrong?

Dividing by  $10^{-17}$  produced a huge number  $10^{17}$   
 $\leadsto$  all errors just exploded!

On a computer, you never want to divide by tiny numbers!

**Solution:** Use the larger pivot (in absolute value)

$$\begin{bmatrix} 10^{-17} & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 10^{-17} & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow 10^{-17} R_1} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 &= 1 \end{aligned} \quad \leadsto \begin{aligned} x_1 &= 1 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

computer:  
 $1 - 10^{-17} \approx 1$   
 $1 - 2 \cdot 10^{-17} \approx 1$

# Gaussian Elimination using Maximal Partial Pivoting:

In step (a), row swap so that the **largest** number in the column (in absolute value) is the pivot.

This is much more **numerically stable**.  
→ avoids dividing by tiny numbers.

## $PA=LU$ Decompositions

**Recall:** LU only works when you don't need to **row swap** when eliminating.

But elimination works much better with row swaps!  
Need to tweak LU.

**Idea:** Do all the row swaps you need **first**, then do elimination without row swaps.

(How do you know in advance which row swaps to do? You don't - need to do more bookkeeping)

**Def:** A **permutation matrix** is a product of elementary matrices for row swaps.

**PA=LU Decomposition:** Any matrix A has a factorization

$$PA = LU$$

Do a bunch of row swaps on A...

... then do an LU decomp

where:

P: permutation matrix

L: lower-unitriangular matrix

U: REF matrix

Eg:

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix} \xrightarrow{\substack{\text{maximal} \\ \text{partial} \\ \text{pivoting}}} R_1 \leftrightarrow R_2 \begin{bmatrix} -10 & -20 & -30 \\ 1 & 1 & 1 \\ 5 & 15 & 10 \end{bmatrix} P_1 \\
 &\xrightarrow{\substack{R_2 + \frac{1}{10}R_1 \\ R_3 + \frac{1}{2}R_1}} \begin{bmatrix} -10 & -20 & -30 \\ 0 & -1 & -2 \\ 0 & 5 & -5 \end{bmatrix} E_1, E_2 \\
 &\xrightarrow{\substack{\text{maximal} \\ \text{partial} \\ \text{pivoting}}} R_2 \leftrightarrow R_3 \begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & -1 & -2 \end{bmatrix} P_2 \\
 &\xrightarrow{R_3 + \frac{1}{5}R_2} \begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix} E_3 \\
 &\quad \quad \quad \underbrace{\hspace{10em}}_U
 \end{aligned}$$

$$U = E_3 P_2 E_2 E_1 P_1 A$$

not lower-unitriangular!

Trick:  $P_2 E_2 E_1 = \underbrace{(P_2 E_2 E_1 P_2)}_{\text{is lower-uniform!}} P_2$   $P_2 P_2 = I_3$

$$P_2 E_2 E_1 P_2 = P_2 E_2 E_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{matrix} R_2 \leftrightarrow R_1 \\ R_3 \leftrightarrow R_1 \end{matrix} P_2 \begin{bmatrix} 1 & 0 & 0 \\ 1/10 & 0 & 1 \\ 1/2 & 1 & 0 \end{bmatrix}$$

$$\begin{matrix} R_1 \leftrightarrow R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/10 & 0 & 1 \end{bmatrix} \quad \text{lower-uniform!}$$

Then  $U = E_3 (P_2 E_2 E_1 P_2) P_1 A$

$$\Rightarrow PA = LU$$

for  $P = P_2 P_1$   $L = (E_3 P_2 E_2 E_1 P_2)^{-1}$

Here is an efficient way of doing the bookkeeping.

# Algorithm (PA=LU Decomposition; 3-column method):

Input: Any matrix  $A$

Output: A factorization  $PA = LU$  where:

$P$ : permutation matrix

$L$ : lower-unitriangular matrix

$U$ : REF matrix

Procedure: Perform Gaussian elimination using any pivoting strategy (eg. maximal partial pivoting). Keep track of row operations as follows: start with a blank matrix  $L$  and an identity matrix  $P$ .

- for each row replacement  $R_i \leftarrow R_i + c R_j$ , put  $-c$  in the  $(i,j)$  entry of  $L$  (as before)
- for each row swap  $R_i \leftrightarrow R_j$ , swap the corresponding rows of  $L$  and  $P$ .

Add 1's & 0's to the remaining blank entries of  $L$  (1's on the diagonal, 0's elsewhere)

Then

$$PA = LU$$

Eg:  $A = \begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix}$

3 columns  
 $P \quad L \quad U$

start with  $I_m$  here  $\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$      $\begin{bmatrix} \text{blank} \end{bmatrix}$      $\begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix}$  ← start with A in the U col

$R_1 \leftrightarrow R_2$   $\rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$      $\begin{bmatrix} \text{blank} \end{bmatrix}$      $\begin{bmatrix} -10 & -20 & -30 \\ 1 & 1 & 1 \\ 5 & 15 & 10 \end{bmatrix}$

$R_2 + \frac{1}{10}R_1$   
 $R_3 + \frac{1}{2}R_1$   $\rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$      $\begin{bmatrix} -1/10 \\ -1/2 \end{bmatrix}$      $\begin{bmatrix} -10 & -20 & -30 \\ 0 & -1 & -2 \\ 0 & 5 & -5 \end{bmatrix}$

$R_2 \leftrightarrow R_3$   $\rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$      $\begin{bmatrix} -1/2 \\ -1/10 \end{bmatrix}$      $\begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & -1 & -2 \end{bmatrix}$

$R_3 + \frac{1}{5}R_2$   $\rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$      $\begin{bmatrix} -1/2 \\ -1/10 & -1/5 \end{bmatrix}$      $\begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix}$

add these 2  
 $L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/10 & -1/5 & 1 \end{bmatrix}$

Check:

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/10 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix}$

Can we still use this to solve  $Ax=b$ ?

## Algorithm (solving $Ax=b$ using $PA=LU$ )

Solving  $Ax=b$  is the same as  $PAx=Pb$ , so:

- (0) Compute  $Pb$  (re-order the entries of  $b$ ) (0 flops)
- (1) Solve  $Ly=Pb$  using forward-substitution ( $n^2-n$  flops)
- (2) Solve  $Ux=y$  using backward-substitution ( $n^2$  flops)

$$\text{Then } PAx = LUx = Ly = Pb$$

$$\Rightarrow Ax=b \quad (\text{multiply both sides by } P^{-1}) \quad \checkmark$$

(total  $\approx n^2$  flops)

Eg: Solve  $Ax=b$  for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$$

$$(0) \quad Pb = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix}$$

$$(1) \quad \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/10 & -1/5 & 1 \end{bmatrix} y = Pb = \begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix} \rightsquigarrow \begin{matrix} y_1 = -10 \\ y_2 = 5 \\ y_3 = 0 \end{matrix}$$

$$(2) \quad \begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix} x = \begin{bmatrix} -10 \\ 5 \\ 0 \end{bmatrix} \rightsquigarrow \begin{matrix} x_1 = -1 \\ x_2 = 1 \\ x_3 = 0 \end{matrix} \quad x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$