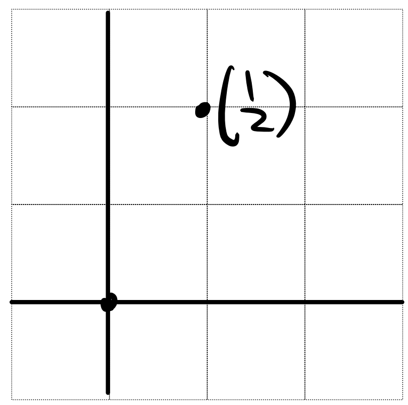
# Geometry of Vectors

**Recall:** A vector in $\mathbb{R}^n$ is a list of $n$ numbers:

$$v = (x_1, \dots, x_n) \in \mathbb{R}^n$$

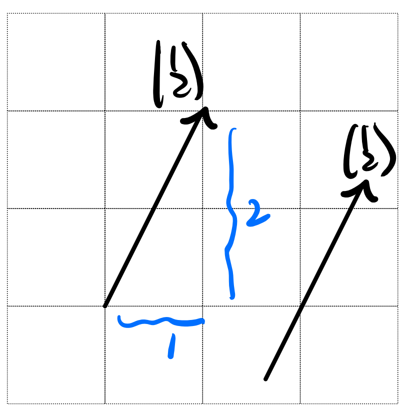
We can draw a vector as a point in Euclidean space:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x\text{-coordinate} \\ y\text{-coordinate} \end{pmatrix}$$




We will often consider a vector as an arrow, or displacement: measures the difference between two points.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x\text{-displacement} \\ y\text{-displacement} \end{pmatrix}$$




NB the tail of the vector can be anywhere, but by default vectors start at 0

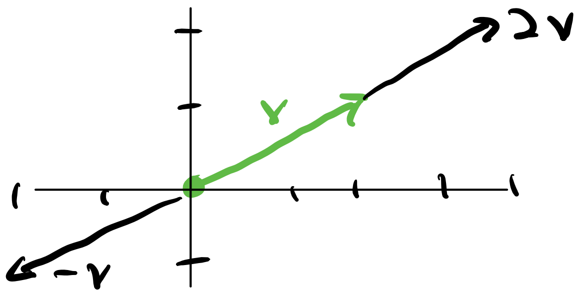
How do algebraic operations behave geometrically?
We'll describe in terms of arrows.

# Scalar Multiplication:

- the length of $cv$ is $|c| \times$ the length of $v$
- the direction of $cv$ is
  - → the same as $v$ if $c>0$
  - → the opposite from $v$ if $c<0$

[demo]

Eg: $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$




$2v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$-v = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

# Vector Addition:

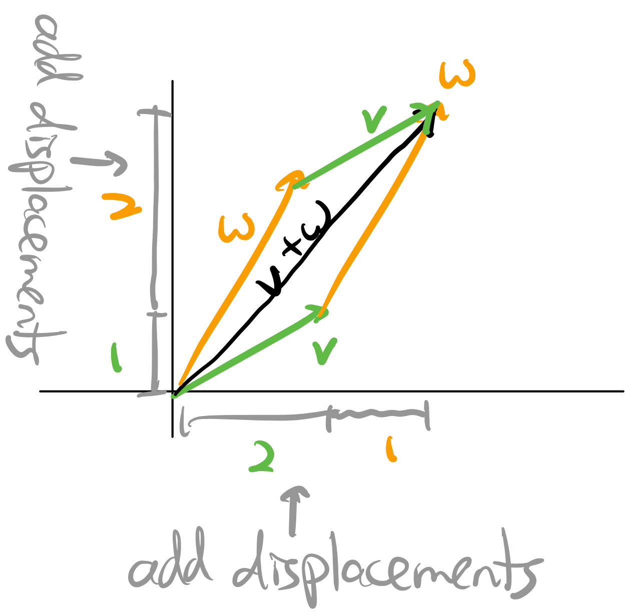
This just adds the displacements.

**Parallelogram Law:** to draw $v+w$, draw the tail of $v$ at the head of $w$ (or vice-versa); the head of $v$ is at $v+w$.

Eg: $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$v+w = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$




[demo]

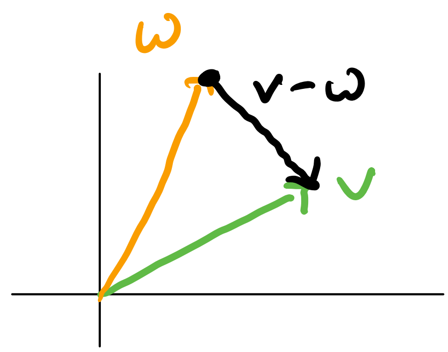
## Vector Subtraction: $w+(v-w)=v$

So $v-w$ starts at the head of $w$ & ends at the head of $v$.

Eg: $v=\begin{pmatrix}2\\1\end{pmatrix}$

$w=\begin{pmatrix}1\\2\end{pmatrix}$

$v-w=\begin{pmatrix}1\\-1\end{pmatrix}$



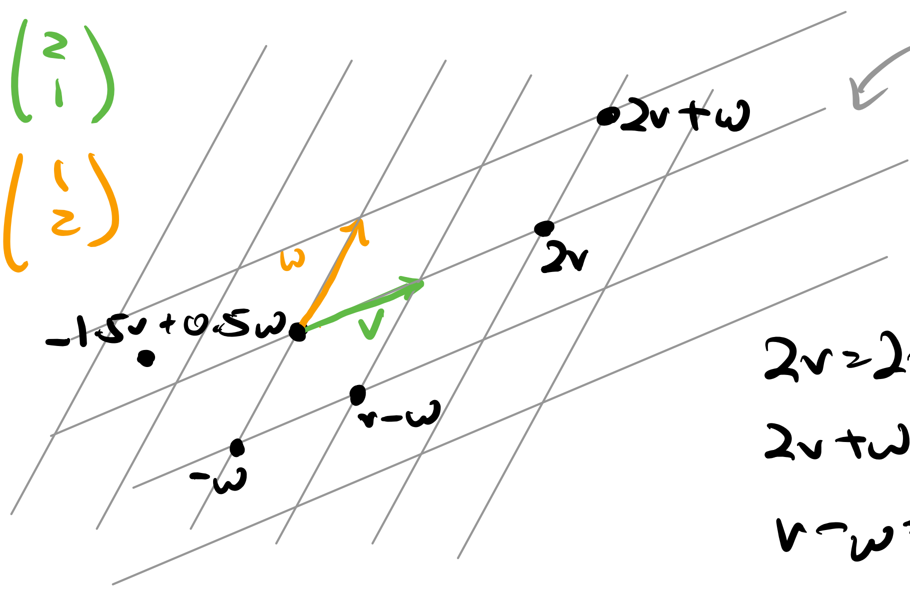

[demo]

## Linear Combinations:

First scale, then add.

Eg: $v=\begin{pmatrix}2\\1\end{pmatrix}$

$w=\begin{pmatrix}1\\2\end{pmatrix}$




grid lines are in v- and w- directions

[demos]

$2v=2v+0w=\begin{pmatrix}4\\2\end{pmatrix}$

$2v+w=\begin{pmatrix}5\\4\end{pmatrix}$

$v-w=\begin{pmatrix}1\\-1\end{pmatrix}$

This is like giving directions: "To get to $-1.5v+0.5w$, first go $1.5\times$ length of $v$ in the opposite $v$-direction, then go $0.5\times$ length of $w$ in the $w$-direction."

# Spans

Look out for two subtle concepts below.

**Recall:** the notion of "all linear combinations of some set of vectors" came up twice last time:

- $Ax=b$ is consistent if
  $$b \in \begin{pmatrix}\text{all linear combinations} \\ \text{of the columns of } A\end{pmatrix}$$
- If so, the solution set of $Ax=b$ is
  $$\begin{pmatrix}\text{particular} \\ \text{solution}\end{pmatrix} + \begin{pmatrix}\text{all linear combinations} \\ \text{of some vectors}\end{pmatrix}$$

**Def:** The span of a list of vectors is the set of all linear combinations of those vectors:

$$\operatorname{Span}\{v_1, v_2, \dots, v_n\} = \underbrace{\{}_{\text{"the set of"}} \underbrace{c_1v_1 + c_2v_2 + \dots + c_nv_n}_{\text{"all things of this form"}} \underbrace{:}_{\text{"such that"}} \underbrace{c_1, \dots, c_n \in \mathbb{R}}_{\text{"these conditions hold"}} \}$$

This is set-builder notation!

**Translation of the above:**

(1) $Ax=b$ is consistent $\iff b \in \operatorname{Span}\{\text{columns of } A\}$

(2) If so, the solution set of $Ax=b$ is
$$\begin{pmatrix}\text{particular} \\ \text{solution}\end{pmatrix} + \operatorname{Span}\{\text{some vectors}\}$$

## Column Picture Criterion for Consistency (again)

$Ax=b$ is consistent (has at least one solution)

$\Updownarrow$

$b \in \text{Span}\{\text{columns of } A\}$

subtle concept #1

## What do spans look like?

It's the smallest "linear space" (line, plane, etc.) containing all your vectors & the origin.

Eg: $\text{Span}\{v\} = \{cv : c \in \mathbb{R}\}$

→ If $v \neq 0$ get the line thru $0$ & $v$

→ $\text{Span}\{0\} = \{c \cdot 0 : c \in \mathbb{R}\} = \{0\}$

= the set containing only $0$

[demo]

Eg: $\text{Span}\{v, w\} = \{cv + dw : c, d \in \mathbb{R}\}$

→ If $v, w$ are not collinear, get the plane defined by $0$, $v$, and $w$

→ If $v, w$ are collinear and nonzero, get the line thru $v, w$, and $0$.

→ If $v = w = 0$ get $\{0\}$

[demo]

Eg: $\operatorname{Span}\{u,v,w\} = \{bu+cv+dw : b,c,d \in \mathbb{R}\}$

- → If $u,v,w$ are not coplanar, get space
- → If $u,v,w$ are coplanar but not collinear, get the plane containing them.
- → If $u,v,w$ are collinear & not all zero, get the line thru $u,v,w$, and $0$.
- → If $u=v=w=0$ get $\{0\}$

[demo]

Eg: $\operatorname{Span}\{\} = \{0\}$ (by convention)

empty set

Warning: Be careful to distinguish these sets:

- $\{\}$: the empty set has no vectors in it at all (eg. the solution set of an inconsistent system)
- $\{0\}$: the point contains (only) the zero vector

The difference is: $\{0\}$ contains $0$; $\{\}$ does not.

Likewise,

- $\{v_1,\ldots,v_n\}$: a set with n vectors in it
- $\operatorname{Span}\{v_1,\ldots,v_n\}$ is a linear space: it contains infinitely many vectors (unless $v_1=\cdots=v_n=0$)

eg. a line 

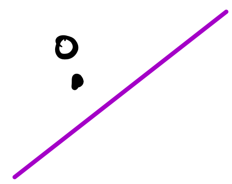
The span construction allows you to parametrically describe a linear space (infinite set) using a finite amount of data.

$\longrightarrow$ Now you can do computations!

**NB:** Every span contains the zero vector!

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$$

So eg. this line is not a span:



Eg: $\{\}$ is not a span! It does not contain 0.

Q: Is $\begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$ in $\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \right\}$ ?

In other words, does

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$$

have a solution?

Let's solve this vector equation:

$$\left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} x_1 = -1 \\ x_2 = -9 \end{array}$$

Answer: yes, $\begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} \in \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \right\}$

This example is just the "$\Longrightarrow$" of the statement:

$Ax=b$ is consistent $\Longleftrightarrow$ $b \in \text{Span}\{\text{cols of } A\}$

## Column Picture Criterion for Consistency:

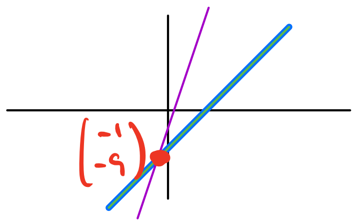
subtle concept #1

- $\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ is consistent because

$$\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \in \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\}$$

[demo]

row picture: $x_1 - x_2 = 8$
$2x_1 - 2x_2 = 16$
$6x_1 - x_2 = 3$

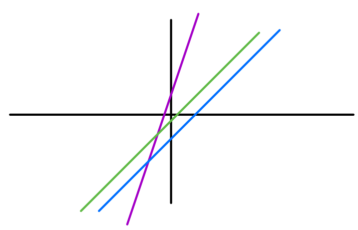



- $\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix}$ is inconsistent because

$$\begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix} \notin \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\}$$

[demo]

row picture: $x_1 - x_2 = 7$
$2x_1 - 2x_2 = 1$
$6x_1 - x_2 = -1$



# Homogeneous Equations

If the solution set of $Ax=b$ is a span

$\Rightarrow 0$ is a solution (every span contains 0)

$\Rightarrow A0=b \Rightarrow b=0$

Let's study this case.

**Def:** $Ax=b$ is called **homogeneous** if $b=0$.

**Eg:**

$$\begin{aligned} x_1+2x_2+2x_3+x_4&=0 \\ 2x_1+4x_2+x_3-x_4&=0 \end{aligned}$$

**NB:** A homogeneous equation is always consistent since $0$ is a solution: $A\cdot 0=0$

**Def:** The **trivial solution** of a homogeneous equation $Ax=0$ is the zero vector.

**Eg:** Let's solve the homogeneous system

$$\begin{aligned} x_1+2x_2+2x_3+x_4&=0 \\ 2x_1+4x_2+x_3-x_4&=0 \end{aligned} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 -= 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \div= -3} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 -= 2R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

PF $\rightsquigarrow$

$$\begin{aligned} x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned}$$

PVF $\rightsquigarrow$ $$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Observations:

(1) The augmented column is always zero.
When solving homogeneous equations, you don't need to write the augmented column.

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array}\right] \rightsquigarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix}$$

(2) The particular solution is the zero vector

(3) The solution set is

$$\mathrm{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Fact: The PVF of a homogeneous system always has particular solution $= 0$. The solution set is the span of the other vectors you've produced.

# Inhomogeneous Equations

**Def:** $Ax=b$ is called inhomogeneous if $\boxed{b\neq 0}$.

What's the difference from homogeneous equations?

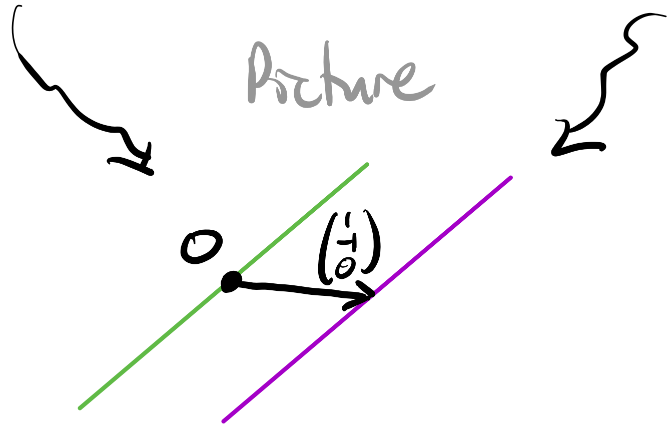
**NB:** It can be inconsistent!

Let's solve the inhomogeneous & homogeneous versions:

**Eg:**

| inhomogeneous | | homogeneous |
|---|---|---|
| $\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ | | $\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ |
| $\downarrow$ | (augmented) matrix | $\downarrow$ |
| $\left[\begin{array}{ccc\|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right]$ | | $\left[\begin{array}{ccc\|c} 2 & 1 & 12 & 0 \\ 1 & 2 & 9 & 0 \end{array}\right]$ |
| $\downarrow$ | RREF | $\downarrow$ |
| $\left[\begin{array}{ccc\|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$ | same | $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}$ |
| $\downarrow$ | PVF | $\downarrow$ |
| $x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$ | | $x = z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$ |
| $\downarrow$ | Solution set | $\downarrow$ |
| $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\}$ | same | $\mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\}$ |

Picture



[demo]

The only difference is the particular solution! Otherwise they're parallel lines.

subtle concept #2

**Facts:**

(1) The solution set of $Ax=0$ is a span.

(2) The solution set of $Ax=b$ is not a span for $b\neq 0$: it is a translate of the solution set of $Ax=0$ by a particular solution. (Or it is empty.)

$$\begin{pmatrix}\text{solutions}\\ \text{of } Ax=0\end{pmatrix} = (\text{zero}) + \text{Span}\left\{\begin{matrix}\text{vectors}\\ \text{from PVF}\end{matrix}\right\}$$

same vectors!

$$\begin{pmatrix}\text{solutions}\\ \text{of } Ax=b\end{pmatrix} = \begin{pmatrix}\text{particular}\\ \text{solution}\end{pmatrix} + \text{Span}\left\{\begin{matrix}\text{vectors}\\ \text{from PVF}\end{matrix}\right\}$$

In fact, to get the solutions of $Ax=b$ you can translate the solutions of $Ax=0$ by any single solution of $Ax=b$.

→ Say $p$ is some solution of $Ax=b$, so $Ap=b$.

Then $Ax=0 \iff Ap+Ax=b \iff A(p+x)=b$

vectors of the form $p+$ (soln of $Ax=0$)

NB: Expressing a solution set as a (translate of a) span means writing it in parametric form:

$$x \in \begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}\right\}$$

$$\iff x = \begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + c\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + d\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

parameters

So think:

Spans $\equiv$ Parametric form

# Row & Column Picture

We now know:

Row Picture

(1) $\begin{pmatrix}\text{All solutions}\\ \text{of } Ax=b\end{pmatrix}$

Span

$= \begin{pmatrix}\text{Some solution}\\ \text{of } Ax=b\end{pmatrix} + \begin{pmatrix}\text{All solutions}\\ \text{of } Ax=0\end{pmatrix}$

subtle concept #2

or is empty. In particular, all nonempty solution sets are parallel and look the same.

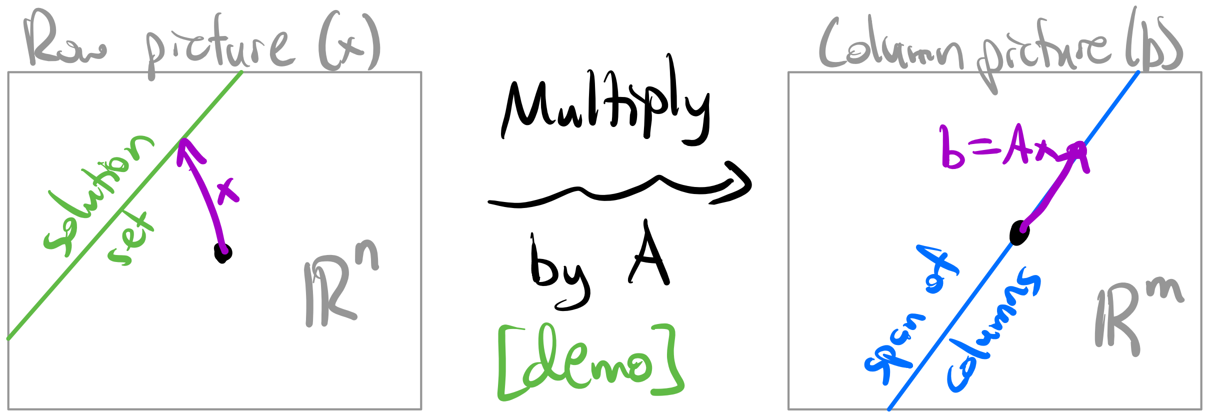
Column Picture

(2) $Ax=b$ is consistent $\iff$ $b$ is in the span of the columns of $A$.

subtle concept #1

We can draw these both at the same time:

subtle concept #2




subtle concept #1

In this picture, we think of $A$ as a function:

$x \in \mathbb{R}^n$ is the input (row picture)

$Ax \in \mathbb{R}^m$ is the output (column picture)

Solving $Ax=b$ means finding all inputs with output $=b$.

The solution set lives in the... row picture!
The b-vectors live in the... column picture!
The columns all live in the... column picture!
That's how you keep them straight.