MATH 218D-1
PRACTICE FINAL EXAMINATION

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic. You may bring a 8.5×11**-inch note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

Problem 1. [15 points]

a) Which one of the following symmetric matrices is positive-definite? Circle the correct answer.

$$\begin{pmatrix} 2 & -4 & 0 & 0 \\ -4 & 11 & 0 & -9 \\ 0 & 0 & 9 & 0 \\ 0 & -9 & 0 & 29 \end{pmatrix} \qquad \begin{pmatrix} 2 & -4 & 0 & 0 \\ -4 & 11 & 0 & -9 \\ 0 & 0 & 4 & 0 \\ 0 & -9 & 0 & 25 \end{pmatrix}$$

b) Find the LU decomposition of the following positive-definite symmetric matrix.

$$A = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 11 & -9 \\ 0 & -9 & 31 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & -4 & 0 \\ 0 & 3 & -9 \\ 0 & 0 & 4 \end{pmatrix}$$

c) Find the $A = LDL^T$ decomposition and Cholesky decomposition $A = L_1 L_1^T$ of the matrix in **b)**.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad L_1 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -2\sqrt{2} & \sqrt{3} & 0 \\ 0 & -3\sqrt{3} & 2 \end{pmatrix}$$

Problem 2. [20 points]

Consider the quadratic form

$$q(x_1, x_2) = 3x_1^2 + 3x_2^2 + 2x_1x_2.$$

a) Find the symmetric matrix S such that $q(x) = x^T S x$.

$$S = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

b) Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

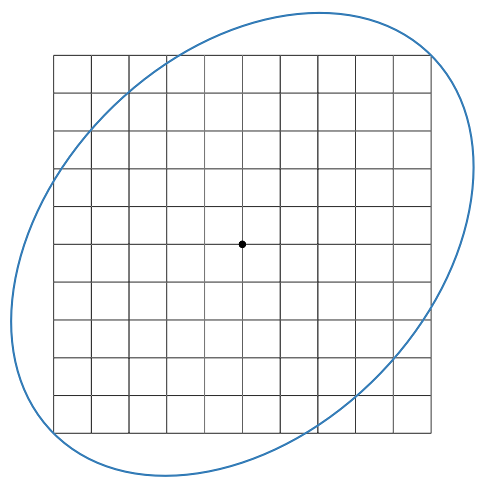
$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

c) Find the minimum and maximum values of $q(x_1, x_2)$ subject to the constraint $x_1^2 + x_2^2 = 1$, and all vectors (x_1, x_2) at which these values are achieved.

Min: $q = 2$ is achieved at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Max: $q = 4$ is achieved at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

d) Draw the ellipse defined by the equation $q(x_1, x_2) = 1$. Grid lines are 0.1 units apart. Be precise!



Problem 3. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

a) Compute the symmetric matrix $S = AA^T$.

$$S = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

b) Find the eigenvalues of S, and find an orthonormal eigenbasis.

Eigenvalues: $4, 2$ Eigenbasis: $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) Compute the singular value decomposition of A in outer product form.

$$A = 2\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} + \sqrt{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

d) Compute the cross product

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

e) Compute the singular value decomposition of A in matrix form.

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}^T$$

Problem 4. [20 points]

Consider the initial value problem

$$\begin{cases} u_1' = u_1 & u_1(0) = 1 \\ u_2' = u_1 + 2u_2 & u_2(0) = 0 \end{cases}$$

a) Find a matrix A such that $u' = Au$, where $u = (u_1, u_2)$.

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

b) Compute the characteristic polynomial of A, and find the eigenvalues.

$p(\lambda) = \lambda^2 - 3\lambda + 2$ Eigenvalues: 1 and 2

c) Find an eigenbasis of A.

$$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

d) Solve the initial value problem.

$$u_1 = e^t$$
$$u_2 = e^{2t} - e^t$$

Problem 5. [20 points]

Consider the subspace V of $\mathbf{R}^4$ defined by the equation

$$x_1 + x_2 + 2x_3 - 12x_4 = 0.$$

a) Compute an *orthogonal* basis for V.

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \\ 1 \end{pmatrix} \right\}$$

b) Compute an *orthogonal* basis for $V^\perp$.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ -12 \end{pmatrix} \right\}$$

c) Compute the projection matrix P_V.

$$P_V = \frac{1}{150} \begin{pmatrix} 149 & -1 & -2 & 12 \\ -1 & 149 & -2 & 12 \\ -2 & -2 & 146 & 24 \\ 12 & 12 & 24 & 6 \end{pmatrix}$$

d) Compute the orthogonal projection of the vector $b = (-1, -1, 4, -12)$ onto V.

$$b_V = \begin{pmatrix} -2 \\ -2 \\ 2 \\ 0 \end{pmatrix}$$

e) The distance from $(-1, -1, 4, -12)$ to V is $\boxed{5\sqrt{6}}$.

Problem 6. [25 points]

The centered data matrix

$$A = \begin{pmatrix} -2.73 & 0.714 & 1.90 & -2.08 & 2.11 & 0.0825 \\ -8.48 & 2.73 & 0.187 & -1.53 & 6.99 & 0.106 \\ -5.63 & 2.03 & -7.30 & 6.37 & 4.71 & -0.179 \\ -4.91 & 1.64 & 3.12 & -3.91 & 3.91 & 0.149 \end{pmatrix}$$

has normalized singular value decomposition

$$\frac{1}{\sqrt{5}}A = 7\begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix}\begin{pmatrix} -0.738 \\ 0.245 \\ -0.164 \\ 0.0479 \\ 0.606 \\ 0.00268 \end{pmatrix}^T + 5\begin{pmatrix} 0.297 \\ 0.260 \\ -0.753 \\ 0.527 \end{pmatrix}\begin{pmatrix} -0.122 \\ 0.0231 \\ 0.694 \\ -0.704 \\ 0.0853 \\ 0.0237 \end{pmatrix}^T + 0.1\begin{pmatrix} 0.867 \\ 0.0176 \\ -0.000602 \\ -0.498 \end{pmatrix}\begin{pmatrix} -0.284 \\ -0.682 \\ 0.465 \\ 0.487 \\ 0.0173 \\ -0.00242 \end{pmatrix}^T + 0.05\begin{pmatrix} -0.350 \\ 0.659 \\ -0.320 \\ -0.585 \end{pmatrix}\begin{pmatrix} 0.380 \\ -0.539 \\ -0.284 \\ -0.285 \\ 0.626 \\ 0.102 \end{pmatrix}^T$$

a) The total variance of the data points is $s^2 = \boxed{74.0125}$.

b) The nonzero eigenvalues of $\frac{1}{5}A^TA$ are $\boxed{49, 25, .01, .0025}$.

c) $\frac{1}{5}AA^T = QDQ^T$ where Q and D are orthogonal and diagonal matrices, respectively:

$$Q = \begin{pmatrix} 0.195 & 0.297 & 0.867 & -0.350 \\ 0.706 & 0.260 & 0.0176 & 0.659 \\ 0.575 & -0.753 & -0.000602 & -0.320 \\ 0.364 & 0.527 & -0.498 & -0.585 \end{pmatrix} \qquad D = \begin{pmatrix} 49 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & .01 & 0 \\ 0 & 0 & 0 & .0025 \end{pmatrix}$$

d) The variance $s(u)^2$ is maximized at the unit vector

$$u = \begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix} \text{ with maximum value } s(u)^2 = 49.$$

e) The line L of best fit is spanned by the vector

$$\begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix} \text{ with } s(L^\perp)^2 = 25.0125$$

f) The orthogonal projection of the first column of A onto L is

$$\begin{pmatrix} -2.25 \\ -8.15 \\ -6.64 \\ -4.20 \end{pmatrix}$$

g) The plane V of best fit is spanned by the vectors

$$\left\{ \begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix}, \begin{pmatrix} 0.297 \\ 0.260 \\ -0.753 \\ 0.527 \end{pmatrix} \right\} \text{ with } s(V^\perp)^2 = 0.0125$$

h) What kind of linear space best describes the shape of the data? Circle one.

line plane 3—space $\mathbf{R}^4$

i) Suppose that the recentered data point $(1, 1, x, y)$ was drawn from the same data set. What would you predict for the values of x and y?
[**Hint:** to maximize your exam score, finish the rest of the exam first.]

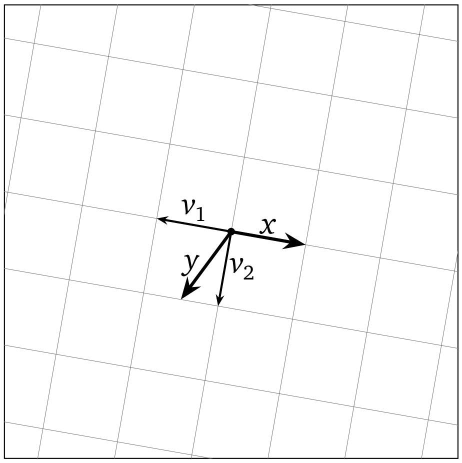
$x = -2.29$
$y = 1.78$

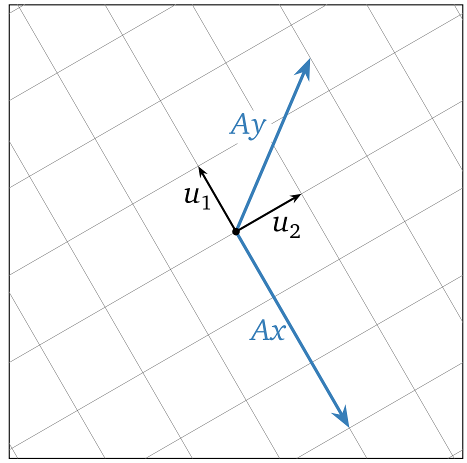
Problem 7. [20 points]

a) A certain 2×2 matrix A has the singular value decomposition

$$A = 3u_1 v_1^T + 2u_2 v_2^T$$

where u_1, u_2, v_1, v_2 are drawn in the diagrams below. Given x and y in the diagram on the left, draw Ax and Ay on the diagram on the right.

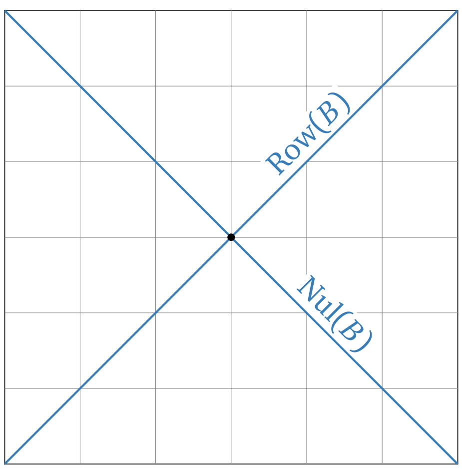


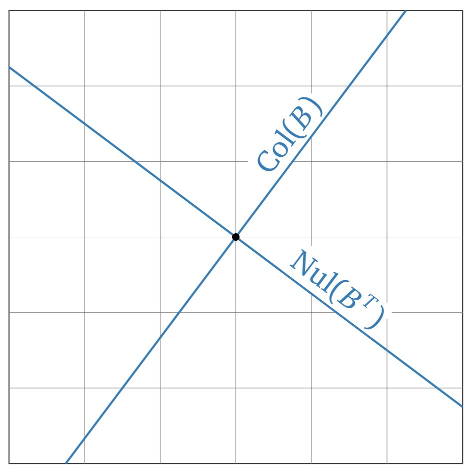


b) A certain 2×2 matrix B has singular value decomposition

$$B = 13 \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Draw and *label* Row(B) and Nul(B) on the grid on the left, and draw and *label* Col(B) and Nul(B^T) on the grid on the right.





Problem 8. [20 points]

True/false problems: **circle** the correct answer. No justification is needed.

All matrices in this problem have real entries.

a) T F If A is a matrix of rank r, then A is a linear combination of r rank-1 matrices.

b) T F For any matrix A, the matrices AA^T and A^TA have the same eigenvalues.

c) T F The only positive-semidefinite projection matrix is the identity.

d) T F Any 3×3 real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

e) T F The eigenvalues of an invertible matrix are all nonzero.

f) T F A matrix with nonzero orthogonal columns has full column rank.

g) T F If P_V is the projection matrix onto a subspace V, then $\mathrm{Nul}(P_V)$ is the orthogonal complement of $\mathrm{Col}(P_V)$.

h) T F If $b \in V^\perp$ then $b_V = b$.

i) T F If A is an $m \times n$ matrix and $\mathrm{Col}(A) = \mathbf{R}^m$, then A has full column rank.

j) T F If U is an echelon form of A, then $\mathrm{Nul}(U) = \mathrm{Nul}(A)$.

Problem 9. [20 points]

Short-answer problems: no justification is necessary.

a) Find the matrix A satisfying

$$A\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix}\qquad A\begin{pmatrix}-2\\1\\0\end{pmatrix}=0\qquad A\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

$$A=\begin{pmatrix}-1&-2&-2\\0&0&1\\0&0&1\end{pmatrix}$$

b) Which of the following sets form a basis for $\mathrm{Span}\left\{\binom{1}{1},\binom{-1}{1},\binom{1}{2}\right\}$?

○ $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$ ● $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\right\}$

● $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\right\}$ ● $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

c) Let A be a 4×10 centered data matrix and let V be a subspace of $\mathbf{R}^4$. Suppose that $s(V^\perp)^2=0$. What does this tell you about the columns of A?
They all lie on V.

d) Let A be a 2×2 matrix that is neither invertible nor diagonalizable. What is the characteristic polynomial of A?

$$p(\lambda)=\lambda^2$$

e) Let A be a 2×3 matrix such that $Ax=\binom{1}{2}$ does not have a solution. Which of the following are *impossible*?

○ The solution set of $Ax=\binom{1}{1}$ is empty.
● The solution set of $Ax=\binom{1}{1}$ is a point.
● The solution set of $Ax=\binom{1}{1}$ is a line.
○ The solution set of $Ax=\binom{1}{1}$ is a plane.
○ The solution set of $Ax=0$ is $\mathbf{R}^3$.
● The solution set of $Ax=\binom{1}{1}$ is a plane and the solution set of $Ax=0$ is $\mathbf{R}^3$.

Problem 10.

[20 points]

Give examples satisfying the following requirements, or explain why no such example exists. (No justification is needed if an example does exist.)

All matrices in this problem have real entries.

a) A 2×2 symmetric matrix S with eigenvalue $\frac{1}{2}(1 + i\sqrt{3})$.
Impossible by the spectral theorem.

b) A 3×2 matrix A such that A^+A is the identity matrix.
Any 3×2 matrix with full column rank will do. For instance,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

c) A nonzero 2×2 matrix A such that $Ax = 0$ is inconsistent.
Impossible: $x = 0$ is a solution.

d) A basis for $\mathbf{R}^3$ containing the vector $(1, 1, 1)$.
There are many correct answers. For instance,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

e) A basis for $\mathbf{R}^3$ containing the vector $(0, 0, 0)$.
Impossible: a set containing the zero vector is linearly dependent.