# MATH 218D-1
# PRACTICE FINAL EXAMINATION

| Name | | Duke NetID | |
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic. You may bring a $8.5 \times 11$**-inch note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

## Problem 1. [15 points]

**a)** Which one of the following symmetric matrices is positive-definite? Circle the correct answer.

$$\begin{pmatrix} 2 & -4 & 0 & 0 \\ -4 & 11 & 0 & -9 \\ 0 & 0 & 9 & 0 \\ 0 & -9 & 0 & 29 \end{pmatrix} \qquad \begin{pmatrix} 2 & -4 & 0 & 0 \\ -4 & 11 & 0 & -9 \\ 0 & 0 & 4 & 0 \\ 0 & -9 & 0 & 25 \end{pmatrix}$$

**b)** Find the $LU$ decomposition of the following positive-definite symmetric matrix.

$$A = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 11 & -9 \\ 0 & -9 & 31 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & -4 & 0 \\ 0 & 3 & -9 \\ 0 & 0 & 4 \end{pmatrix}$$

**c)** Find the $A = LDL^T$ decomposition and Cholesky decomposition $A = L_1 L_1^T$ of the matrix in **b)**.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad L_1 = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -2\sqrt{2} & \sqrt{3} & 0 \\ 0 & -3\sqrt{3} & 2 \end{pmatrix}$$

## Problem 2. [20 points]

Consider the quadratic form

$$q(x_1, x_2) = 3x_1^2 + 3x_2^2 + 2x_1x_2.$$

**a)** Find the symmetric matrix $S$ such that $q(x) = x^T Sx$.

$$S = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

**b)** Find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $S = QDQ^T$.

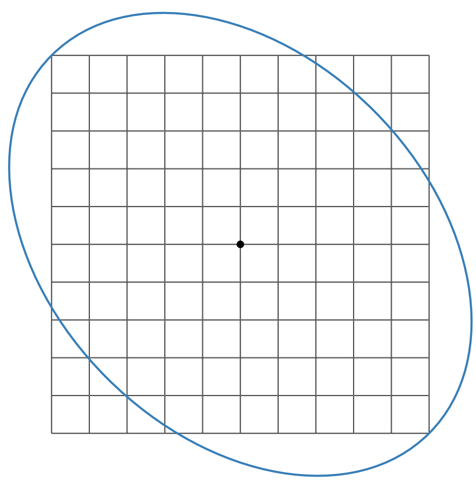
$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

**c)** Find the minimum and maximum values of $q(x_1, x_2)$ subject to the constraint $x_1^2 + x_2^2 = 1$, and all vectors $(x_1, x_2)$ at which these values are achieved.

Min: $q = 2$ is achieved at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Max: $q = 4$ is achieved at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

**d)** Draw the ellipse defined by the equation $q(x_1, x_2) = 1$. Grid lines are 0.1 units apart. Be precise!



## Problem 3. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

**a)** Compute the symmetric matrix $S = AA^T$.

$$S = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

**b)** Find the eigenvalues of $S$, and find an orthonormal eigenbasis.

Eigenvalues: $4, 2$ Eigenbasis: $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**c)** Compute the singular value decomposition of $A$ in outer product form.

$$A = 2\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} + \sqrt{2}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

**d)** Compute the cross product

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

**e)** Compute the singular value decomposition of $A$ in matrix form.

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}^T$$

## Problem 4.

[20 points]

Consider the initial value problem

$$\begin{cases} u_1' = u_1 & u_1(0) = 1 \\ u_2' = u_1 + 2u_2 & u_2(0) = 0 \end{cases}$$

**a)** Find a matrix $A$ such that $u' = Au$, where $u = (u_1, u_2)$.

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

**b)** Compute the characteristic polynomial of $A$, and find the eigenvalues.

$p(\lambda) = \lambda^2 - 3\lambda + 2$ Eigenvalues: 1 and 2

**c)** Find an eigenbasis of $A$.

$$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

**d)** Solve the initial value problem.

$$u_1 = e^t$$
$$u_2 = e^{2t} - e^t$$

## Problem 5. [20 points]

Consider the subspace $V$ of $\mathbf{R}^4$ defined by the equation

$$x_1 + x_2 + 2x_3 - 12x_4 = 0.$$

**a)** Compute an *orthogonal* basis for $V$.

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \\ 1 \end{pmatrix} \right\}$$

**b)** Compute an *orthogonal* basis for $V^\perp$.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ -12 \end{pmatrix} \right\}$$

**c)** Compute the projection matrix $P_V$.

$$P_V = \frac{1}{150} \begin{pmatrix} 149 & -1 & -2 & 12 \\ -1 & 149 & -2 & 12 \\ -2 & -2 & 146 & 24 \\ 12 & 12 & 24 & 6 \end{pmatrix}$$

**d)** Compute the orthogonal projection of the vector $b = (-1, -1, 4, -12)$ onto $V$.

$$b_V = \begin{pmatrix} -2 \\ -2 \\ 2 \\ 0 \end{pmatrix}$$

**e)** The distance from $(-1, -1, 4, -12)$ to $V$ is $\boxed{5\sqrt{6}}$.

## Problem 6. [25 points]

The centered data matrix

$$A = \begin{pmatrix} -2.73 & 0.714 & 1.90 & -2.08 & 2.11 & 0.0825 \\ -8.48 & 2.73 & 0.187 & -1.53 & 6.99 & 0.106 \\ -5.63 & 2.03 & -7.30 & 6.37 & 4.71 & -0.179 \\ -4.91 & 1.64 & 3.12 & -3.91 & 3.91 & 0.149 \end{pmatrix}$$

has normalized singular value decomposition

$$\frac{1}{\sqrt{5}}A = 7\begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix}\begin{pmatrix} -0.738 \\ 0.245 \\ -0.164 \\ 0.0479 \\ 0.606 \\ 0.00268 \end{pmatrix}^T + 5\begin{pmatrix} 0.297 \\ 0.260 \\ -0.753 \\ 0.527 \end{pmatrix}\begin{pmatrix} -0.122 \\ 0.0231 \\ 0.694 \\ -0.704 \\ 0.0853 \\ 0.0237 \end{pmatrix}^T + 0.1\begin{pmatrix} 0.867 \\ 0.0176 \\ -0.000602 \\ -0.498 \end{pmatrix}\begin{pmatrix} -0.284 \\ -0.682 \\ 0.465 \\ 0.487 \\ 0.0173 \\ -0.00242 \end{pmatrix}^T + 0.05\begin{pmatrix} -0.350 \\ 0.659 \\ -0.320 \\ -0.585 \end{pmatrix}\begin{pmatrix} 0.380 \\ -0.539 \\ -0.284 \\ -0.285 \\ 0.626 \\ 0.102 \end{pmatrix}^T$$

**a)** The total variance of the data points is $s^2 = \boxed{74.0125}$.

**b)** The nonzero eigenvalues of $\frac{1}{5}A^TA$ are $\boxed{49,\ 25,\ .01,\ .0025}$.

**c)** $\frac{1}{5}AA^T = QDQ^T$ where $Q$ and $D$ are orthogonal and diagonal matrices, respectively:

$$Q = \begin{pmatrix} 0.195 & 0.297 & 0.867 & -0.350 \\ 0.706 & 0.260 & 0.0176 & 0.659 \\ 0.575 & -0.753 & -0.000602 & -0.320 \\ 0.364 & 0.527 & -0.498 & -0.585 \end{pmatrix} \qquad D = \begin{pmatrix} 49 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & .01 & 0 \\ 0 & 0 & 0 & .0025 \end{pmatrix}$$

**d)** The variance $s(u)^2$ is maximized at the unit vector

$$u = \begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix} \text{ with maximum value } s(u)^2 = 49.$$

**e)** The line $L$ of best fit is spanned by the vector

$$\begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix} \text{ with } s(L^\perp)^2 = 25.0125$$

**f)** The orthogonal projection of the first column of $A$ onto $L$ is

$$\begin{pmatrix} -2.25 \\ -8.15 \\ -6.64 \\ -4.20 \end{pmatrix}$$

**g)** The plane $V$ of best fit is spanned by the vectors

$$\left\{ \begin{pmatrix} 0.195 \\ 0.706 \\ 0.575 \\ 0.364 \end{pmatrix}, \begin{pmatrix} 0.297 \\ 0.260 \\ -0.753 \\ 0.527 \end{pmatrix} \right\} \text{ with } s(V^\perp)^2 = 0.0125$$

**h)** What kind of linear space best describes the shape of the data? Circle one.

line    plane    3−space    $\mathbf{R}^4$

**i)** Suppose that the recentered data point $(1, 1, x, y)$ was drawn from the same data set. What would you predict for the values of $x$ and $y$?
[**Hint:** to maximize your exam score, finish the rest of the exam first.]

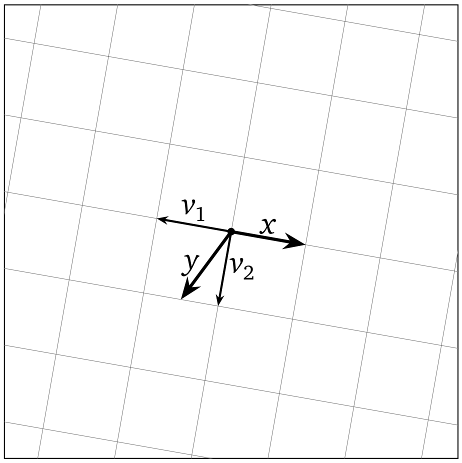
$x = -2.29$
$y = \ \ 1.78$

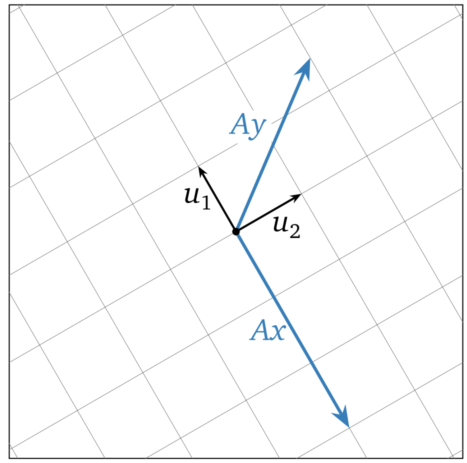
## Problem 7. [20 points]

**a)** A certain $2 \times 2$ matrix $A$ has the singular value decomposition

$$A = 3u_1v_1^T + 2u_2v_2^T$$

where $u_1, u_2, v_1, v_2$ are drawn in the diagrams below. Given $x$ and $y$ in the diagram on the left, draw $Ax$ and $Ay$ on the diagram on the right.

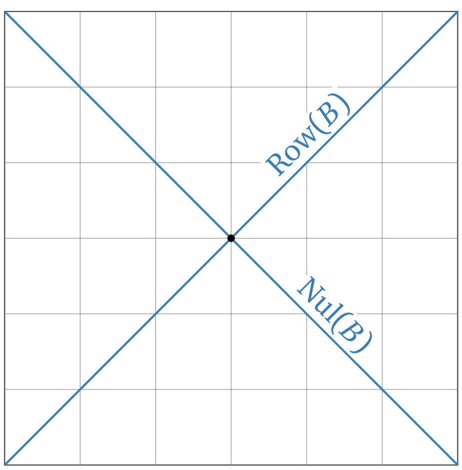


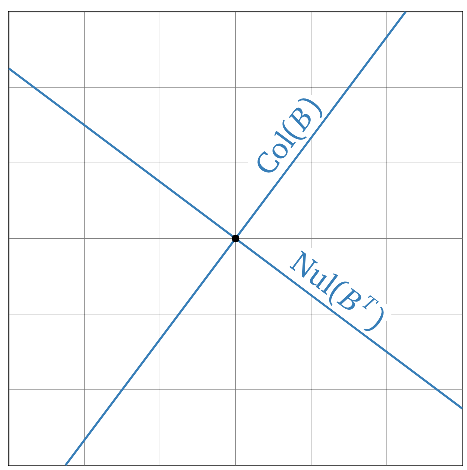




**b)** A certain $2 \times 2$ matrix $B$ has singular value decomposition

$$B = 13\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Draw and *label* Row($B$) and Nul($B$) on the grid on the left, and draw and *label* Col($B$) and Nul($B^T$) on the grid on the right.



## Problem 8. [20 points]

True/false problems: **circle** the correct answer. No justification is needed.

*All matrices in this problem have real entries.*

a) T F If $A$ is a matrix of rank $r$, then $A$ is a linear combination of $r$ rank-1 matrices.

b) T F For any matrix $A$, the matrices $AA^T$ and $A^TA$ have the same eigenvalues.

c) T F The only positive-semidefinite projection matrix is the identity.

d) T F Any $3 \times 3$ real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

e) T F The eigenvalues of an invertible matrix are all nonzero.

f) T F A matrix with nonzero orthogonal columns has full column rank.

g) T F If $P_V$ is the projection matrix onto a subspace $V$, then $\mathrm{Nul}(P_V)$ is the orthogonal complement of $\mathrm{Col}(P_V)$.

h) T F If $b \in V^\perp$ then $b_V = b$.

i) T F If $A$ is an $m \times n$ matrix and $\mathrm{Col}(A) = \mathbf{R}^m$, then $A$ has full column rank.

j) T F If $U$ is an echelon form of $A$, then $\mathrm{Nul}(U) = \mathrm{Nul}(A)$.

## Problem 9.

[20 points]

Short-answer problems: no justification is necessary.

**a)** Find the matrix $A$ satisfying

$$A\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix}\qquad A\begin{pmatrix}-2\\1\\0\end{pmatrix}=0\qquad A\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

$$A=\begin{pmatrix}-1&-2&-2\\0&0&1\\0&0&1\end{pmatrix}$$

**b)** Which of the following sets form a basis for $\mathrm{Span}\left\{\binom{1}{1},\binom{-1}{1},\binom{1}{2}\right\}$?

○ $\left\{\binom{1}{1},\binom{-1}{1},\binom{1}{2}\right\}$ ● $\left\{\binom{1}{1},\binom{1}{2}\right\}$ ○ $\left\{\binom{1}{1},\binom{-1}{-1}\right\}$

● $\left\{\binom{1}{1},\binom{-1}{1}\right\}$ ● $\left\{\binom{1}{0},\binom{0}{1}\right\}$ ○ $\left\{\binom{1}{1},\binom{0}{0}\right\}$ ○ $\left\{\binom{1}{1}\right\}$

**c)** Let $A$ be a $4\times 10$ centered data matrix and let $V$ be a subspace of $\mathbf{R}^4$. Suppose that $s(V^{\perp})^2=0$. What does this tell you about the columns of $A$?
They all lie on $V$.

**d)** Let $A$ be a $2\times 2$ matrix that is neither invertible nor diagonalizable. What is the characteristic polynomial of $A$?

$$p(\lambda)=\lambda^2$$

**e)** Let $A$ be a $2\times 3$ matrix such that $Ax=\binom{1}{2}$ does not have a solution. Which of the following are *impossible*?

○ The solution set of $Ax=\binom{1}{1}$ is empty.
● The solution set of $Ax=\binom{1}{1}$ is a point.
● The solution set of $Ax=\binom{1}{1}$ is a line.
○ The solution set of $Ax=\binom{1}{1}$ is a plane.
○ The solution set of $Ax=0$ is $\mathbf{R}^3$.
● The solution set of $Ax=\binom{1}{1}$ is a plane and the solution set of $Ax=0$ is $\mathbf{R}^3$.

## Problem 10. [20 points]

Give examples satisfying the following requirements, or explain why no such example exists. (No justification is needed if an example does exist.)

*All matrices in this problem have real entries.*

**a)** A $2 \times 2$ symmetric matrix $S$ with eigenvalue $\frac{1}{2}(1 + i\sqrt{3})$.
Impossible by the spectral theorem.

**b)** A $3 \times 2$ matrix $A$ such that $A^{+}A$ is the identity matrix.
Any $3 \times 2$ matrix with full column rank will do. For instance,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

**c)** A nonzero $2 \times 2$ matrix $A$ such that $Ax = 0$ is inconsistent.
Impossible: $x = 0$ is a solution.

**d)** A basis for $\mathbf{R}^3$ containing the vector $(1, 1, 1)$.
There are many correct answers. For instance,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

**e)** A basis for $\mathbf{R}^3$ containing the vector $(0, 0, 0)$.
Impossible: a set containing the zero vector is linearly dependent.