MATH 218D-1
PRACTICE MIDTERM EXAMINATION 1

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

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Problem 1. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 6 & 2 \\ 1 & 3 & 2 \end{pmatrix}.$$

a) Find a $PA = LU$ decomposition of A.

$$P = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \quad L = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \quad U = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

[Scratch work for Problem 1]

(Problem 1, continued)

b) Solve $Ax = b$ for $b = (1, 8, 4)$ using your answer to **a)**.

$$x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

c) Compute A^{-1}. Please write the row operations you performed.

$$A^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

d) Solve $Ax = b$ for an unknown vector $b = (b_1, b_2, b_3)$. Your answer will be a formula in terms of b_1, b_2, b_3.

$$x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 1]

e) Express A^{-1} as a product of elementary matrices. (Write *matrices*, not row operations.)

$A^{-1} =$

f) Express A as a product of elementary matrices.

$A =$

[Scratch work for Problem 1]

Problem 2.

[15 points]

Consider the matrix equation $Ax = b$ for

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 3 & 9 & -2 & 8 \\ 2 & 6 & 2 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

a) Find the parametric vector form of the solution set of $Ax = b$.

$$x = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} +$$

b) Compute the following quantities:

$\text{rank}(A) = \square$ $\quad \dim \text{Nul}(A) = \square$ $\quad \dim \text{Col}(A) = \square$

$\dim \text{Row}(A) = \square$ $\quad \dim \text{Nul}(A^T) = \square$.

c) Find a basis for $\text{Nul}(A)$.

$$\left\{ \qquad \qquad \right\}$$

[Scratch work for Problem 2]

(Problem 2, continued)

d) Given that

$$A\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\2\\8\end{pmatrix},$$

express the solution set of $Ax = (2, 2, 8)$ as a translate of a span.

$$x = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} + \text{Span}\left\{ \qquad\qquad \right\}$$

e) Find a basis for Col(A).

$$\left\{ \qquad\qquad \right\}$$

[Scratch work for Problem 2]

Problem 3. [15 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 1 \\ -7 \\ -7 \end{pmatrix}.$$

a) Find a linear relation among v_1, v_2, v_3.

$$\boxed{}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \boxed{}\begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} + \boxed{}\begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix} = 0$$

b) Span$\{v_1, v_2, v_3\}$ is a (circle one) $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\boxed{}}$.

c) Is $v_4 \in \text{Span}\{v_1, v_2, v_3\}$? If so, express v_4 as a linear combination of v_1, v_2, v_3.

$$\begin{pmatrix} \text{Yes} \\ \text{No} \end{pmatrix} \qquad v_4 = \boxed{}\, v_1 + \boxed{}\, v_2 + \boxed{}\, v_3$$

[Scratch work for Problem 3]

(Problem 3, continued)

d) Is $\{v_1, v_2, v_3, v_4\}$ linearly dependent? If so, find a linear relation among v_1, v_2, v_3, v_4.

$\begin{pmatrix}\text{Yes}\\ \text{No}\end{pmatrix}$ $\quad \boxed{}\, v_1 + \boxed{}\, v_2 + \boxed{}\, v_3 + \boxed{}\, v_4 = 0$

e) $\dim \operatorname{Span}\{v_1, v_2, v_3, v_4\} = \boxed{}$.

f) Which of the following sets form a basis for $\mathbf{R}^3$? Circle all that apply.

$\{v_1, v_2\} \qquad \{v_1, v_2, v_3\} \qquad \{v_1, v_3, v_4\}$

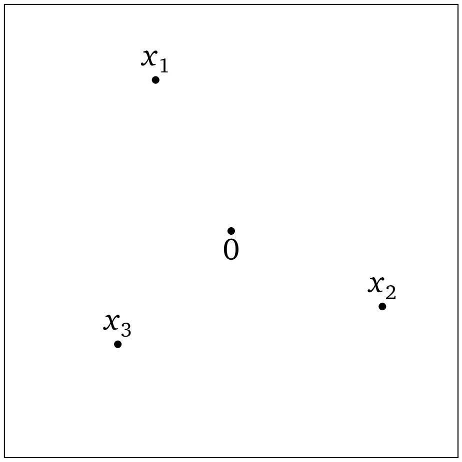
$\{v_2, v_3, v_4\} \qquad \{v_3, v_4\} \qquad \{v_1, v_2, v_3, v_4\}$

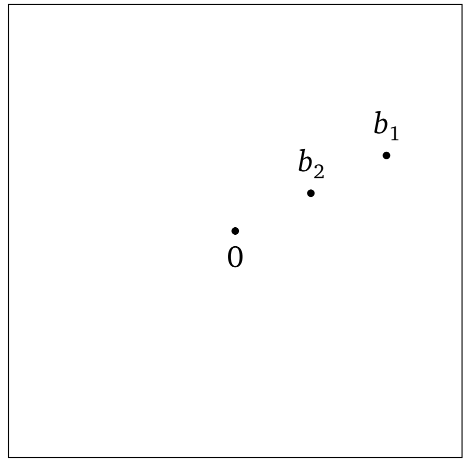
[Scratch work for Problem 3]

Problem 4.

[10 points]

For a certain 2×2 matrix A and vectors $x_1, x_2, x_3 \in \mathbf{R}^2$ drawn on the left, the vectors $b_1 = Ax_1$ and $b_2 = Ax_2 = Ax_3$ are drawn on the right. (All vectors are drawn as points.)





In what follows, it is important that Ax_2 is equal to Ax_3.

a) Draw the solution set of $Ax = b_2$ on the picture on the left.

b) Draw the solution set of $Ax = b_1$ on the picture on the left.

c) Draw Nul(A) on the picture on the left.

d) rank(A) = ☐.

e) Draw Col(A) on the picture on the right.

[Scratch work for Problem 4]

Problem 5. [20 points]

Short-answer questions: no justification is necessary.

a) Consider the matrix

$$A = \begin{pmatrix} 3 & 7 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Which of the following are true about A? Fill in the bubbles of all that apply.

- ◯ A has full row rank.
- ◯ A has full column rank.
- ◯ A is invertible.
- ◯ $\mathrm{Nul}(A) = \{0\}$.
- ◯ $\mathrm{Col}(A) = \mathbf{R}^3$.
- ◯ There exists $b \in \mathbf{R}^3$ such that $Ax = b$ is inconsistent.
- ◯ There exists $b \in \mathbf{R}^3$ such that the solution set of $Ax = b$ is a point.
- ◯ The columns of A are linearly independent.
- ◯ The rows of A are linearly independent.

b) Which of the following are subspaces of $\mathbf{R}^3$? Fill in the bubbles of all that apply.

- ◯ $\mathrm{Nul}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix}$
- ◯ $\mathrm{Col}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix}$
- ◯ $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$
- ◯ The solution set of
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 2 \end{aligned}$$
- ◯ $\{(x, y, z) \in \mathbf{R}^3 \ : \ xyz = 0\}$

c) Find a basis for the left null space of the matrix

$$A = \begin{pmatrix} 3 & 7 & 4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

d) If A is a 4×5 matrix, then

$$\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A) = \boxed{} \qquad \dim \mathrm{Nul}(A) + \dim \mathrm{Row}(A) = \boxed{}.$$

[Scratch work for Problem 5]

Problem 6. [20 points]

Find examples of the following things. If an example exists, no justification is needed; otherwise, explain why no example exists.

a) A 3×3 matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$.

b) Row-equivalent 2×2 matrices A and B with $\mathrm{Col}(A) \neq \mathrm{Col}(B)$.

c) A 3×2 matrix with full row rank.

d) A 2×2 matrix A such that $Ax = 0$ is inconsistent.

[Scratch work for Problem 6]