MATH 218D-1
MIDTERM EXAMINATION 1

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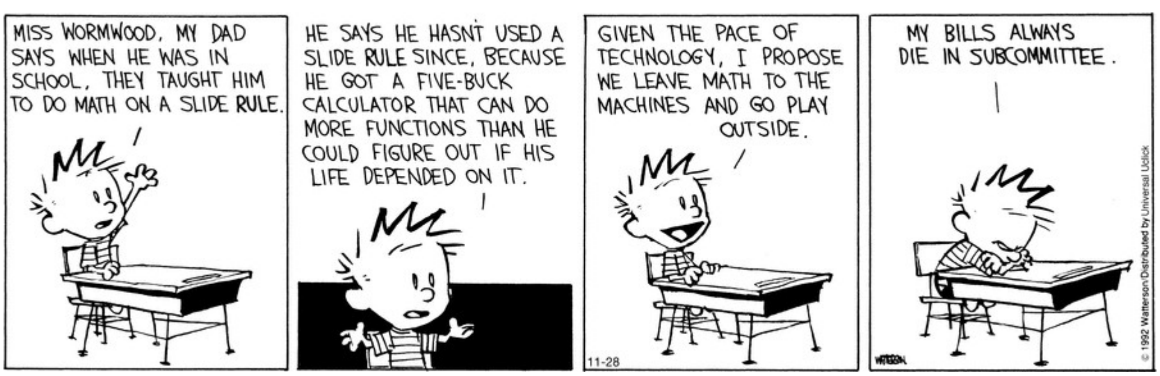
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3 × 5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1. [15 points]

a) Find the LU decomposition of this matrix:

$$A = \begin{pmatrix} -2 & 2 & 1 \\ 4 & -1 & 1 \\ -6 & 12 & 11 \end{pmatrix}.$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} -2 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

b) Express the matrix L that you computed above as a product of three elementary matrices.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

c) Compute L^{-1}.

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -7 & -2 & 1 \end{pmatrix}$$

d) Explain why a computer would probably compute a $PA = LU$ decomposition, beginning with the row swap $R_1 \longleftrightarrow R_3$.

It would choose the largest pivot (in absolute value) to minimize rounding error.

e) Given the decomposition

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & -8 & -5 \\ -1 & -5 & 2 & 0 \\ 2 & 0 & 3 & 2 \\ -1 & -3 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 & 0 & -1 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

solve the equation

$$\begin{pmatrix} 0 & 2 & -8 & -5 \\ -1 & -5 & 2 & 0 \\ 2 & 0 & 3 & 2 \\ -1 & -3 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ -7 \\ 2 \\ -4 \end{pmatrix}.$$

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Problem 2. [20 points]

a) Compute the reduced row echelon form of the matrix

$$\begin{pmatrix} 1 & 3 & 4 & 1 \\ -3 & -9 & -6 & -1 \\ 2 & 6 & 2 & 1 \end{pmatrix}.$$

Be sure to write down all row operations that you perform.

RREF: $\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Now we switch matrices to avoid carry-through error. Consider the matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -1 & 4 & -10 & 1 \\ -3 & 3 & -1 & -3 & -1 \\ 2 & -2 & 2 & -2 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

b) Circle all of the free variables in the system $Ax = 0$:

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$$

c) Compute a basis for Nul(A).

basis: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$

d) Given the identity

$$A \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -15 \\ 12 \end{pmatrix},$$

write the solution set of $Ax = (10, -15, 12)$ as a translate of a span.

solution set: $\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$

e) Compute a basis for Row(A).

basis: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

f) Compute a basis for Col(A).

$$\text{basis: } \left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

g) Compute a basis for Col(A) consisting of vectors with all coordinates equal to 0 or 1.

$$\text{basis: } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

h) Compute a basis for Nul(A^T).

$$\text{basis: } \{\}$$

Problem 3. [15 points]

The matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 5 \\ 2 & -1 & -7 \end{pmatrix} \qquad \text{has null space} \qquad \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

a) Find a linear relation among the columns of A.

$$\boxed{2}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \boxed{-3}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \boxed{1}\begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} = 0$$

b) $\mathrm{rank}(A) = \boxed{2}$

c) Which of the following sets form a basis for Col(A)? Circle all that apply.

$$\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

d) Row(A) is a (circle one) point / line / plane / space in $\mathbf{R}^{\boxed{3}}$.

e) Find a basis for Row(A)$^\perp$. basis: $\left\{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right\}$

f) Which of the following sets form a basis for Nul(A^T)? Circle all that apply.

$$\left\{\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}\right\}, \quad \{\}$$

g) Find a basis for Nul(A^T)$^\perp$. basis: $\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}$

Problem 4.

[10 points]

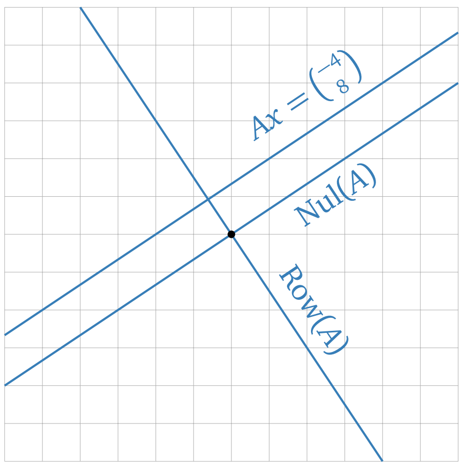
Consider the matrix $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$.

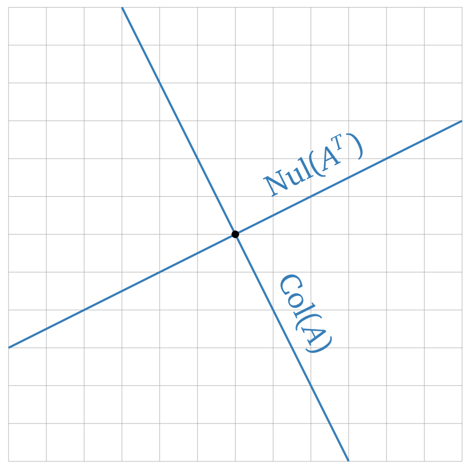
a) Compute bases for all four fundamental subspaces of A.

$$\text{Col}(A)\colon \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \quad \text{Row}(A)\colon \left\{ \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right\}$$

$$\text{Nul}(A^T)\colon \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \quad \text{Nul}(A)\colon \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$$

b) Draw *and label* Row(A) and Nul(A) in the grid on the left, and Col(A) and Nul(A^T) in the grid on the right. Be precise!





c) Draw the solution set of $Ax = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$ in the grid on the left.

Problem 5. [20 points]

Short-answer questions: no justification is necessary unless indicated otherwise.

a) If A is a 5×2 matrix *with full column rank*, which of the following statements must be true about A? Fill in the bubbles of all that apply.

- ○ $\text{rank}(A) = 5$
- ● $\text{Col}(A)$ is a plane in $\mathbf{R}^5$
- ○ $\text{Nul}(A) = \{\}$
- ○ $Ax = b$ has a unique solution for every $b \in \mathbf{R}^5$
- ● $Ax = 0$ has a unique solution
- ○ $\text{Nul}(A^T)$ is a plane in $\mathbf{R}^5$
- ● $\text{Row}(A) = \mathbf{R}^2$

b) A certain 3×3 matrix A has null space equal to $\text{Span}\{(1,1,1)\}$. Which of the following sets is *necessarily* equal to the solution set of $Ax = b$ for *some* vector $b \in \mathbf{R}^3$? Fill in the bubbles of all that apply.

- ● $\text{Span}\{(1,1,1)\}$
- ● $\{\}$
- ○ $\{(1,1,1)\}$
- ○ $\{(t,t,1) \,:\, t \in \mathbf{R}\}$
- ● $\{(t,t,1+t) \,:\, t \in \mathbf{R}\}$
- ● $(11,\, 2,\, -1) + \text{Span}\{(1,1,1)\}$

c) Is this set a subspace?

$$V = \left\{(x,y,z) \in \mathbf{R}^3 \,:\, x^2 + z^2 = 0\right\}$$

If so, express V as the null space or the column space of a matrix. If not, explain why not.

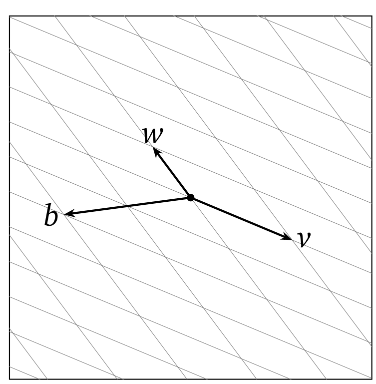
$$V = \left\{(x,y,z) \in \mathbf{R}^3 \,:\, x = z = 0\right\} = \text{Nul}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d) A certain 2×2 matrix

$$A = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix}$$

has columns v and w, pictured below. Solve the equation $Ax = b$, where b is the vector in the picture.

$$x = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$



Problem 6. [20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A 3×3 matrix whose row space and null space are both planes in $\mathbf{R}^3$.

Impossible: if A has 3 columns then $\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A) = 3$.

b) A nonzero 2×2 matrix whose column space is contained in its null space.

For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

c) A 3×3 matrix A such that $\dim \mathrm{Col}(A) = \dim \mathrm{Nul}(A)$.

Impossible: if A has 3 columns then $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = 3$.

d) A 3×3 matrix of rank 2 whose null space is equal to its left null space.

Any symmetric matrix of rank 2 will work. For instance,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$