MATH 218D-1
MIDTERM EXAMINATION 1

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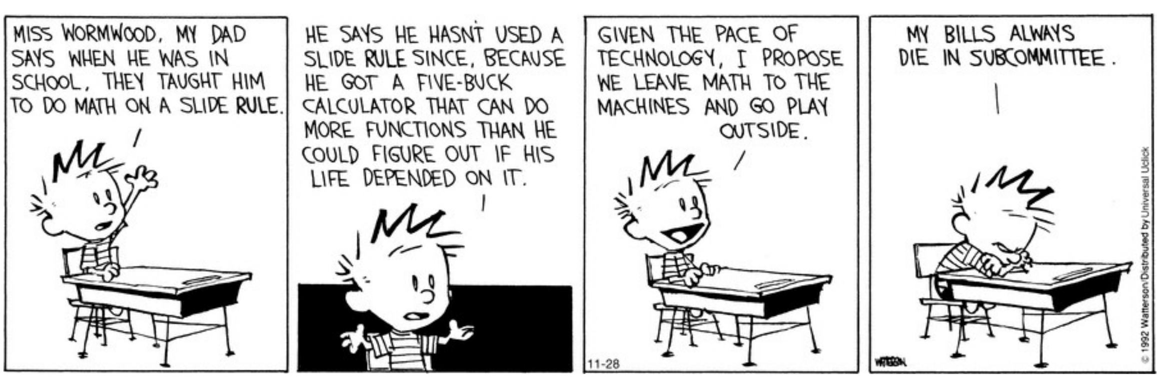
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



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Problem 1. [15 points]

a) Find the LU decomposition of this matrix:

$$A = \begin{pmatrix} -2 & 2 & 1 \\ 4 & -1 & 1 \\ -6 & 12 & 11 \end{pmatrix}.$$

$$L = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \qquad U = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

b) Express the matrix L that you computed above as a product of three elementary matrices.

$$L = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}\begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

[Scratch work for Problem 1]

(Problem 1, continued)

c) Compute L^{-1}.

$$L^{-1} = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

d) Explain why a computer would probably compute a $PA = LU$ decomposition, beginning with the row swap $R_1 \longleftrightarrow R_3$.

e) Given the decomposition

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & -8 & -5 \\ -1 & -5 & 2 & 0 \\ 2 & 0 & 3 & 2 \\ -1 & -3 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 & 0 & -1 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

solve the equation

$$\begin{pmatrix} 0 & 2 & -8 & -5 \\ -1 & -5 & 2 & 0 \\ 2 & 0 & 3 & 2 \\ -1 & -3 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ -7 \\ 2 \\ -4 \end{pmatrix}.$$

$$x = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 1]

Problem 2. [20 points]

a) Compute the reduced row echelon form of the matrix

$$\begin{pmatrix} 1 & 3 & 4 & 1 \\ -3 & -9 & -6 & -1 \\ 2 & 6 & 2 & 1 \end{pmatrix}.$$

Be sure to write down all row operations that you perform.

RREF: $\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$

Now we switch matrices to avoid carry-through error. Consider the matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -1 & 4 & -10 & 1 \\ -3 & 3 & -1 & -3 & -1 \\ 2 & -2 & 2 & -2 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

b) Circle all of the free variables in the system $Ax = 0$:

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$$

c) Compute a basis for Nul(A).

basis: $\left\{ \quad \right\}$

[Scratch work for Problem 2]

d) Given the identity

$$A\begin{pmatrix}1\\-3\\1\\0\\2\end{pmatrix} = \begin{pmatrix}10\\-15\\12\end{pmatrix},$$

write the solution set of $Ax = (10, -15, 12)$ as a translate of a span.

$$\text{solution set: } \begin{pmatrix} \\ \\ \end{pmatrix} + \text{Span}\left\{ \qquad \right\}$$

e) Compute a basis for Row(A).

$$\text{basis: } \left\{ \qquad \right\}$$

f) Compute a basis for Col(A).

$$\text{basis: } \left\{ \qquad \right\}$$

g) Compute a basis for Col(A) consisting of vectors with all coordinates equal to 0 or 1.

$$\text{basis: } \left\{ \qquad \right\}$$

h) Compute a basis for Nul(A^T).

$$\text{basis: } \left\{ \qquad \right\}$$

[Scratch work for Problem 2]

Problem 3. [15 points]

The matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 1 & 5 \\ 2 & -1 & -7 \end{pmatrix} \qquad \text{has null space} \qquad \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

a) Find a linear relation among the columns of A.

$$\boxed{} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \boxed{} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \boxed{} \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} = 0$$

b) $\mathrm{rank}(A) = \boxed{}$

c) Which of the following sets form a basis for $\mathrm{Col}(A)$? Circle all that apply.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

d) $\mathrm{Row}(A)$ is a (circle one) point / line / plane / space in $\mathbf{R}^{\boxed{}}$.

e) Find a basis for $\mathrm{Row}(A)^{\perp}$. basis: $\left\{ \right\}$

f) Which of the following sets form a basis for $\mathrm{Nul}(A^T)$? Circle all that apply.

$$\left\{ \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} \right\}, \quad \{\}$$

g) Find a basis for $\mathrm{Nul}(A^T)^{\perp}$. basis: $\left\{ \right\}$

[Scratch work for Problem 3]

Problem 4.

[10 points]

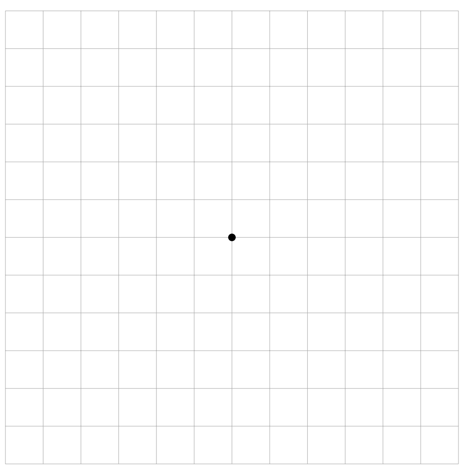
Consider the matrix $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$.

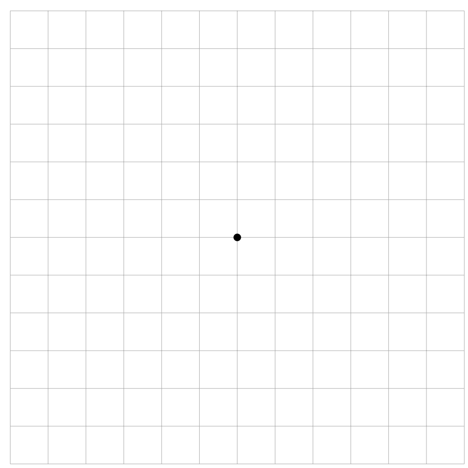
a) Compute bases for all four fundamental subspaces of A.

$$\mathrm{Col}(A): \left\{ \qquad\qquad \right\} \qquad \mathrm{Row}(A): \left\{ \qquad\qquad \right\}$$

$$\mathrm{Nul}(A^T): \left\{ \qquad\qquad \right\} \qquad \mathrm{Nul}(A): \left\{ \qquad\qquad \right\}$$

b) Draw *and label* Row(A) and Nul(A) in the grid on the left, and Col(A) and Nul(A^T) in the grid on the right. Be precise!





c) Draw the solution set of $Ax = \binom{-4}{8}$ in the grid on the left.

[Scratch work for Problem 4]

Problem 5. [20 points]

Short-answer questions: no justification is necessary unless indicated otherwise.

a) If A is a 5×2 matrix *with full column rank*, which of the following statements must be true about A? Fill in the bubbles of all that apply.

◯ $\operatorname{rank}(A) = 5$
◯ $\operatorname{Col}(A)$ is a plane in $\mathbf{R}^5$
◯ $\operatorname{Nul}(A) = \{\}$
◯ $Ax = b$ has a unique solution for every $b \in \mathbf{R}^5$
◯ $Ax = 0$ has a unique solution
◯ $\operatorname{Nul}(A^T)$ is a plane in $\mathbf{R}^5$
◯ $\operatorname{Row}(A) = \mathbf{R}^2$

b) A certain 3×3 matrix A has null space equal to $\operatorname{Span}\{(1,1,1)\}$. Which of the following sets is *necessarily* equal to the solution set of $Ax = b$ for *some* vector $b \in \mathbf{R}^3$? Fill in the bubbles of all that apply.

◯ $\operatorname{Span}\{(1,1,1)\}$
◯ $\{\}$
◯ $\{(1,1,1)\}$
◯ $\{(t,t,1) \,:\, t \in \mathbf{R}\}$
◯ $\{(t,t,1+t) \,:\, t \in \mathbf{R}\}$
◯ $(11,\,2,\,-1) + \operatorname{Span}\{(1,1,1)\}$

c) Is this set a subspace?

$$V = \left\{(x,y,z) \in \mathbf{R}^3 \,:\, x^2 + z^2 = 0\right\}$$

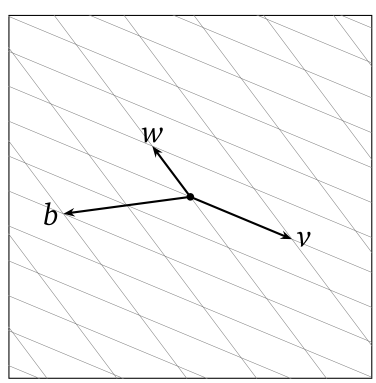
If so, express V as the null space or the column space of a matrix. If not, explain why not.

d) A certain 2×2 matrix

$$A = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix}$$

has columns v and w, pictured below. Solve the equation $Ax = b$, where b is the vector in the picture.

$$x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$



[Scratch work for Problem 5]

Problem 6.

[20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A 3×3 matrix whose row space and null space are both planes in $\mathbf{R}^3$.

b) A nonzero 2×2 matrix whose column space is contained in its null space.

c) A 3×3 matrix A such that $\dim \mathrm{Col}(A) = \dim \mathrm{Nul}(A)$.

d) A 3×3 matrix of rank 2 whose null space is equal to its left null space.

[Scratch work for Problem 6]