# MATH 218D-1
# MIDTERM EXAMINATION 2

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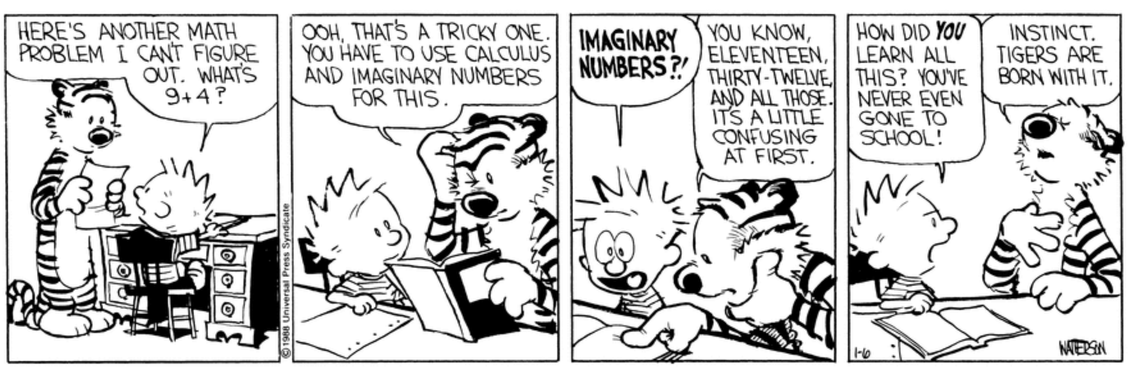
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3 × 5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



## Problem 1. [18 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ -2 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

**a)** The row space of $A$ is a (circle one) $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\boxed{5}}$.

The row space is spanned by the two rows of $A$. The rows are not collinear, so they span a plane, and each row has 5 entries, so they are vectors in $\mathbf{R}^5$.

**b)** Compute the orthogonal projection of $b = (3,0,0,0,-1)$ onto Row($A$).

$$b_{\text{Row}(A)} = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \\ -1 \end{pmatrix}.$$

The easiest way to do this is by solving the normal equation $AA^T x = Ab$. (Note that $\text{Row}(A) = \text{Col}(A^T)$.)

**c)** Compute the orthogonal projection of $b = (3,0,0,0,-1)$ onto Nul($A$).

$$b_{\text{Nul}(A)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\text{Nul}(A) = \text{Row}(A)^\perp$, if your answers to **b)** and **c)** sum to $b$, you'll get full credit.

Now consider the matrix

$$B = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ -2 & 2 & 0 & -4 & -2 \end{pmatrix}.$$

**d)** The row space of $B$ is a (circle one) $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\boxed{5}}$.

This is exactly the same as **a)**, except in this case, the rows *are* collinear, so they span a line.

**e)** Compute the orthogonal projection of $b = (2,0,0,3,-1)$ onto Row($B$).

$$b_{\text{Row}(B)} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Use the formula for projection onto a line: $b_{\text{Row}(B)} = \frac{b \cdot v}{v \cdot v} v$. You can take $v$ to be either row of $B$.

**f)** Compute the projection matrix $P_V$ for $V = \mathrm{Nul}(B)$.

$$P_V = \frac{1}{7}\begin{pmatrix} 6 & 1 & 0 & -2 & -1 \\ 1 & 6 & 0 & 2 & 1 \\ 0 & 0 & 7 & 0 & 0 \\ -2 & 2 & 0 & 3 & -2 \\ -1 & 1 & 0 & -2 & 6 \end{pmatrix}.$$

Since $V^\perp = \mathrm{Row}(B)$ is a line, it's easy to compute $P_{V^\perp} = \frac{vv^T}{v \cdot v}$; then $P_V = I_5 - P_{V^\perp}$.

**g)** Find a basis for $\mathrm{Nul}(P_V)$.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

We have $\mathrm{Nul}(P_V) = V^\perp = \mathrm{Nul}(B)^\perp = \mathrm{Row}(B)$, so you just have to write a multiple of either row of $B$.

## Problem 2. [17 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 4 & -1 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}.$$

Applying the Gram–Schmidt procedure to its columns gives:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 5 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

**a)** Compute the $QR$ decomposition of $A$.

$$Q = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \qquad R = \begin{pmatrix} 2 & 6 & 4 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$

(Check your work! Does $A = QR$? Does $Q$ have orthonormal columns? The rest of the problem will be much easier if so.)

**b)** Find the least-squares solution of $Ax = (2, 0, -4, 2)$.

$$\widehat{x} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$$

It's easiest to solve $R\widehat{x} = Q^T b$ by back-substitution.

**c)** Compute the orthogonal projection of $b = (2, 0, -4, 2)$ onto $V = \mathrm{Col}(A)$.

$$b_V = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 0 \end{pmatrix}.$$

If you multiply $A$ by your answer to **b)** you get full credit.

**d)** Find a nonzero vector $v$ in $\mathrm{Nul}(A^T)$.

$$v = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

We have $\mathrm{Nul}(A^T) = \mathrm{Col}(A)^\perp$, so you just have to subtract your answer to **c)** from $b$.

**e)** Compute the projection matrix $P_V$ onto $V = \mathrm{Col}(A)$. $P_V = \frac{1}{4}\begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix}$

It's probably eaisest to compute $P_V = QQ^T$ since you already know $Q$.

**f)** Find an eigenbasis for $P_V$. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

The 1-eigenspace of $P_V$ is $V = \mathrm{Col}(A)$, and the columns of $A$ (or of $Q$) form a basis for $\mathrm{Col}(A)$. This gives you 3 basis vectors, so you need one more. The 0-eigenspace is $V^\perp = \mathrm{Nul}(A^T)$, and you found a basis for that in **d)**. Combining these gives an eigenbasis.

## Problem 3.

[15 points]

The matrix

$$A = \begin{pmatrix} 61/2 & 12 & -7/2 \\ -51 & -20 & 6 \\ 75 & 30 & -8 \end{pmatrix}$$

has eigenvectors

$$w_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \quad w_3 = \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}.$$

**a)** Find the eigenvalue associated to each of these eigenvectors.

$$\lambda_1 = \boxed{-\tfrac{1}{2}} \qquad \lambda_2 = \boxed{1} \qquad \lambda_3 = \boxed{2}$$

Just compute $Aw_i$ and see what scalar you have to multiply $w_i$ by to get $Aw_i$. In fact, since you're told that $w_1, w_2, w_3$ are eigenvectors, you really just need to compute the first coordinate of $Aw_i$. For instance,

$$Aw_1 = \begin{pmatrix} -1/2 \\ ? \\ ? \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},$$

so $\lambda_1 = -1/2$.

**b)** Compute the characteristic polynomial of $A$. (You need not expand a product of polynomials.)

$$p(\lambda) = -(\lambda + \tfrac{1}{2})(\lambda - 1)(\lambda - 2)$$

Its linear factors are $\lambda - \lambda_1$, $\lambda - \lambda_2$, $\lambda - \lambda_3$, but its $\lambda^3$-coefficient is $(-1)^3 = -1$.

**c)** Find an invertible matrix $C$ and a diagonal matrix $D$ such that $A = CDC^{-1}$.

$$C = \begin{pmatrix} 1 & 1 & -2 \\ -2 & -1 & 3 \\ 2 & 5 & -6 \end{pmatrix} \quad D = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The columns of $C$ are $w_1, w_2, w_3$ and the diagonal entries of $D$ are your answers to **a)**.

**d)** If $v = (-1, 3, 2)$, compute $A^{100}v$. (You can write your answer in terms of $w_1, w_2, w_3$.)

$$A^{100}v = -(-\tfrac{1}{2})^{100}w_1 + 2w_2 + 2^{100}w_3$$

First you have to expand in the eigenbasis: you solve $v = x_1w_1 + x_2w_2 + x_3w_3$ to get $x_1 = -1$, $x_2 = 2$, $x_3 = 1$ so that $v = -w_1 + 2w_2 + w_3$. Then multiply by $A^{100}$.

**e)** For which vectors $u$ does $\|A^k u\|$ *not* approach $\infty$ as $k \to \infty$?

When you expand $u = x_1w_1 + x_2w_2 + x_3w_3$ in the eigenbasis, you get

$$A^n u = \left(-\frac{1}{2}\right)^n x_1w_1 + x_2w_2 + 2^{100}x_3w_3.$$

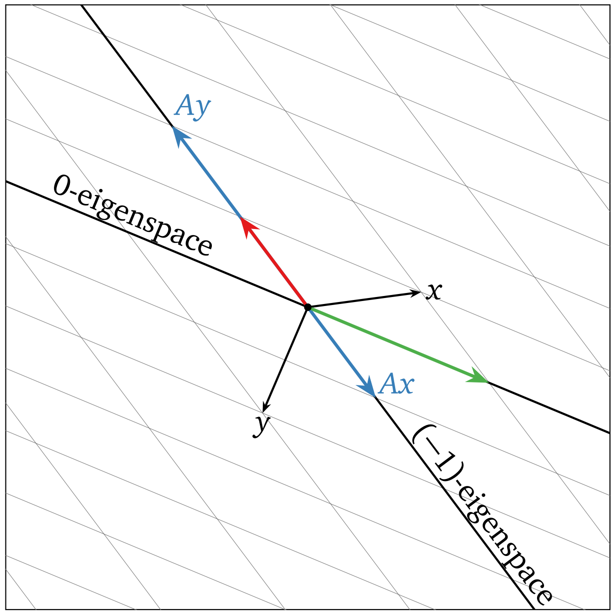
If $x_3 = 0$ then this approaches $x_2w_2$; otherwise it becomes arbitrarily long. So the answer is "all $u \in \text{Span}\{w_1, w_2\}$".

## Problem 4.

[10 points]

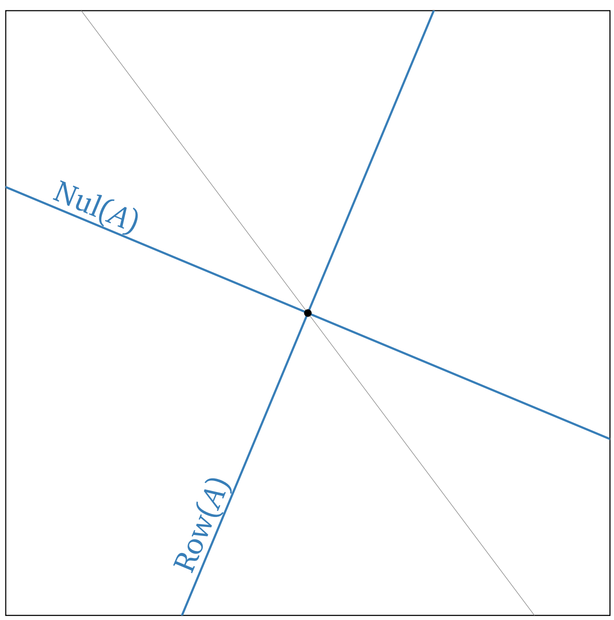
A certain $2 \times 2$ matrix $A$ has eigenvalues 0 and $-1$, with corresponding eigenspaces drawn below.

**a)** Draw and label $Ax$ and $Ay$.




Write $x$ as the sum of the red vector and the green vector. When you multiply by $A$, the green vector goes to 0 (it's in the null space), and the red vector gets negated. Likewise for $y$.

**b)** Draw and label Nul($A$) and Row($A$). (The eigenspaces are reproduced in gray.)



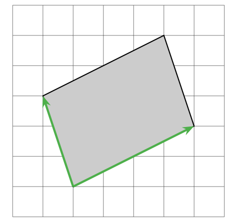

The null space is the 0-eigenspace. The row space is its orthogonal complement.

## Problem 5. [20 points]

Short-answer questions: no explanation is needed unless indicated otherwise.

**a)** Compute the area of the parallelogram. (Grid marks are one unit apart.)

This is the paralellogram determined by the vectors $\binom{-1}{3}$ and $\binom{4}{2}$ (in green), so the area is $\left|\det\begin{pmatrix}-1 & 3\\ 4 & 2\end{pmatrix}\right|$.



area = 14

**b)** For which value(s) of $k$, if any, is the following matrix not invertible?

Compute $\det(A)$ by expanding cofactors. It helps to do the row operation $R_3 \mathrel{-}= 2R_1$ first.

$$A = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & k & 4 \\ 2 & 1 & -1 & 2 \\ 0 & 3 & 2 & 0 \end{pmatrix}$$

$k =$ 16

**c)** Suppose that $A$ is an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2.$$

Which of the following can you determine from this information?

- ● The number $n$.
- ● The trace of $A$.
- ● The determinant of $A$.
- ● The eigenvalues of $A$.
- ● Whether $A$ is invertible.
- ○ Whether $A$ is diagonalizable.

The number $n$ is the degree of $p(\lambda)$; $\mathrm{Tr}(A)$ is the $\lambda^2$-coefficient; $\det(A)$ is the constant coefficient; the eigenvalues are the roots; $A$ is invertible if $\det(A) \neq 0$. You don't know if $A$ is diagonalizable because GM(2) could be 1 or 2.

**d)** Suppose that $v$ is a 3-eigenvector of $A$. Briefly explain why $v \in \mathrm{Col}(A)$.

If $Av = 3v$ then $v = A(\frac{1}{3}v)$.

## Problem 6.

[20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

**a)** A $2 \times 2$ *non-diagonalizable* matrix with eigenvalues 1 and $-1$.

Impossible: a $2 \times 2$ matrix with two eigenvalues is diagonalizable.

**b)** A $2 \times 2$ matrix whose 1-eigenspace is the line $x + 2y = 0$ and whose 2-eigenspace is the line $x + 3y = 0$.

$$CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

**c)** A $3 \times 2$ matrix $A$ and a vector $b$ such that $Ax = b$ does not have a least-squares solution.

Impossible: $Ax = b$ always has a least-squares solution.

**d)** A $2 \times 2$ matrix that is *orthogonal* but has no zero entries.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for most values of $\theta$.