MATH 218D-1
MIDTERM EXAMINATION 2

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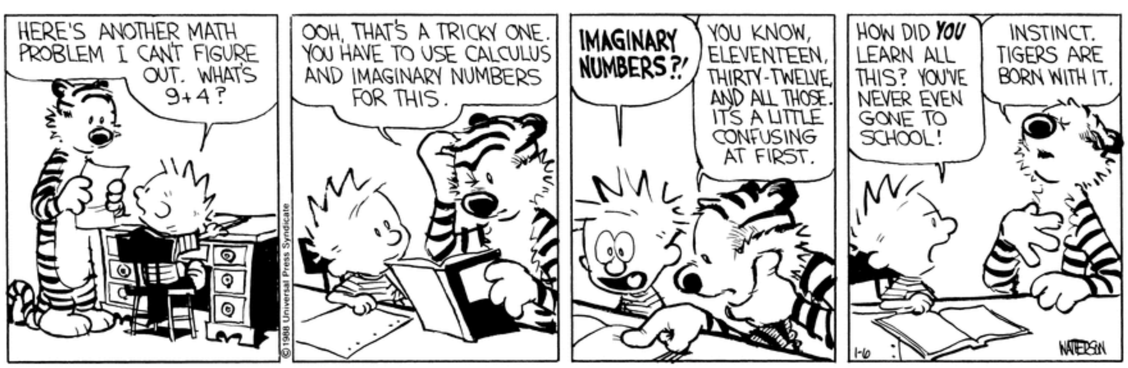
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



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Problem 1. [18 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ -2 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

a) The row space of A is a (circle one) $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\square}$.

b) Compute the orthogonal projection of $b = (3, 0, 0, 0, -1)$ onto Row(A).

$$b_{\text{Row}(A)} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

c) Compute the orthogonal projection of $b = (3, 0, 0, 0, -1)$ onto Nul(A).

$$b_{\text{Nul}(A)} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

[Scratch work for Problem 1]

(Problem 1, continued)

Now consider the matrix

$$B = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ -2 & 2 & 0 & -4 & -2 \end{pmatrix}.$$

d) The row space of B is a (circle one) $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\square}$.

e) Compute the orthogonal projection of $b = (2, 0, 0, 3, -1)$ onto Row(B).

$$b_{\text{Row}(B)} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

f) Compute the projection matrix P_V for $V = \text{Nul}(B)$.

$$P_V = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

g) Find a basis for Nul(P_V).

$$\left\{ \quad \right\}.$$

[Scratch work for Problem 1]

Problem 2.

[17 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 4 & -1 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}.$$

Applying the Gram–Schmidt procedure to its columns gives:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 5 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

a) Compute the QR decomposition of A.

$$Q = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

$$R = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

(Check your work! Does $A = QR$? Does Q have orthonormal columns? The rest of the problem will be much easier if so.)

b) Find the least-squares solution of $Ax = (2, 0, -4, 2)$.

$$\widehat{x} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 2]

c) Compute the orthogonal projection of $b = (2, 0, -4, 2)$ onto $V = \mathrm{Col}(A)$.

$$b_V = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

d) Find vector v in $\mathrm{Nul}(A^T)$.

$$v = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

e) Compute the projection matrix P_V onto $V = \mathrm{Col}(A)$.

$$P_V = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}.$$

f) Find an eigenbasis for P_V.

$$\left\{ \quad \right\}.$$

[Scratch work for Problem 2]

Problem 3.

[15 points]

The matrix

$$A = \begin{pmatrix} 61/2 & 12 & -7/2 \\ -51 & -20 & 6 \\ 75 & 30 & -8 \end{pmatrix}$$

has eigenvectors

$$w_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \quad w_3 = \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}.$$

a) Find the eigenvalue associated to each of these eigenvectors.

$$\lambda_1 = \boxed{} \qquad \lambda_2 = \boxed{} \qquad \lambda_3 = \boxed{}$$

b) Compute the characteristic polynomial of A. (You need not expand a product of polynomials.)

$$p(\lambda) =$$

c) Find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

d) If $v = (-1, 3, 2)$, compute $A^{100}v$. (You can write your answer in terms of w_1, w_2, w_3.)

$$A^{100}v =$$

e) For which vectors u does $||A^k u||$ *not* approach ∞ as $k \to \infty$?

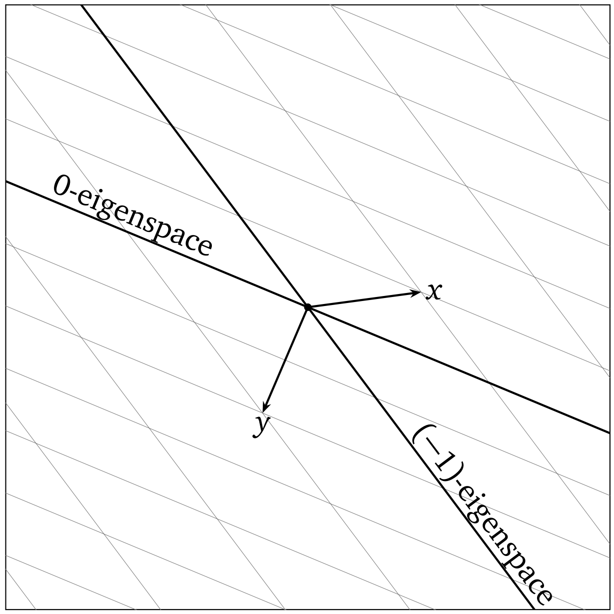
[Scratch work for Problem 3]

Problem 4.

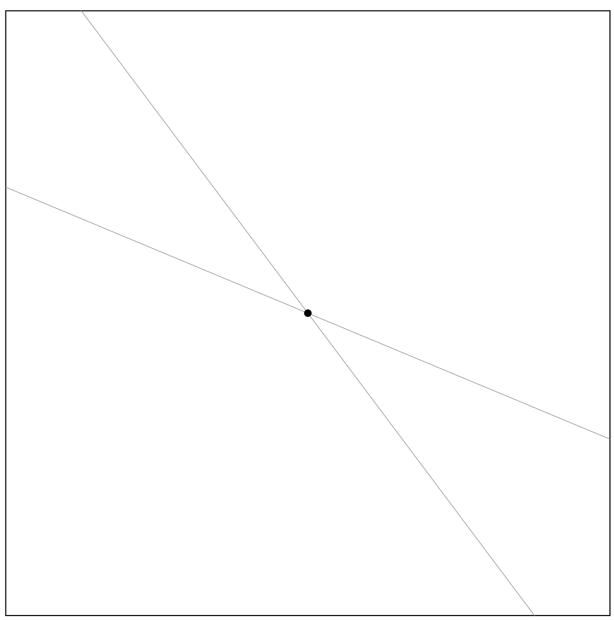
[10 points]

A certain 2×2 matrix A has eigenvalues 0 and -1, with corresponding eigenspaces drawn below.

a) Draw and label Ax and Ay.



b) Draw and label Nul(A) and Row(A). (The eigenspaces are reproduced in gray.)

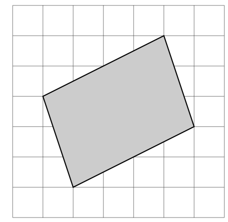


[Scratch work for Problem 4]

Problem 5. [20 points]

Short-answer questions: no explanation is needed unless indicated otherwise.

a) Compute the area of the parallelogram. (Grid marks are one unit apart.)



area = ☐

b) For which value(s) of k, if any, is the following matrix not invertible?

$$A = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & k & 4 \\ 2 & 1 & -1 & 2 \\ 0 & 3 & 2 & 0 \end{pmatrix}$$

$k =$ ☐

c) Suppose that A is an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2.$$

Which of the following can you determine from this information?

- ◯ The number n.
- ◯ The trace of A.
- ◯ The determinant of A.
- ◯ The eigenvalues of A.
- ◯ Whether A is invertible.
- ◯ Whether A is diagonalizable.

d) Suppose that v is a 3-eigenvector of A. Briefly explain why $v \in \mathrm{Col}(A)$.

[Scratch work for Problem 5]

Problem 6. [20 points]

In each part, either provide an example, or explain why no example exists. (No explanation is required if an example does exist.)

a) A 2×2 *non-diagonalizable* matrix with eigenvalues 1 and -1.

b) A 2×2 matrix whose 1-eigenspace is the line $x + 2y = 0$ and whose 2-eigenspace is the line $x + 3y = 0$.

c) A 3×2 matrix A and a vector b such that $Ax = b$ does not have a least-squares solution.

d) A 2×2 matrix that is *orthogonal* but has no zero entries.

[Scratch work for Problem 6]