Math 218D-1: Homework #1

due Wednesday, September 6, at 11:59pm

1. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u + v + w$ and $u + 2v - w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Explain why every linear combination of u, v, w is also a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to ____?

e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

2. Find two *different* triples (x, y, z) such that

$$x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -2 \end{pmatrix} + z \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

How many such triples are there?

3. Decide if each statement is true or false, and explain why.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

c) If v, w are two vectors in $\mathbf{R}^2$, then any other vector b in $\mathbf{R}^2$ is a linear combination of v and w.

4. Suppose that v and w are *unit vectors*: that is, $v \cdot v = 1$ and $w \cdot w = 1$. Compute the following dot products using the algebra of dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v + w) \cdot (v - w)$ **c)** $(v + 2w) \cdot (v - 2w)$.

5. Two vectors v and w are *orthogonal* if $v \cdot w = 0$. Find nonzero vectors v and w in $\mathbf{R}^3$ that are orthogonal to $(1, 1, 1)$ and to each other.

6. Compute the following matrix-vector products using *both* the by-row and by-column methods. If the product is not defined, explain why.

$$\begin{pmatrix}2\\5\end{pmatrix}\begin{pmatrix}1\\-3\\-1\end{pmatrix} \qquad \begin{pmatrix}1&-2\\0&-1\\3&2\end{pmatrix}\begin{pmatrix}1\\0\\-2\end{pmatrix} \qquad \begin{pmatrix}7&2&4\\3&-3&1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}7&4\\-2&2\\4&1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix} \qquad \begin{pmatrix}2&6&-1\end{pmatrix}\begin{pmatrix}5\\-1\\0\end{pmatrix} \qquad \begin{pmatrix}5\\-1\\0\end{pmatrix}\begin{pmatrix}2&6&-1\end{pmatrix}$$

7. Suppose that $u = (x, y, z)$ and $v = (a, b, c)$ are vectors satisfying $2u + 3v = 0$. Find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix}x&a\\y&b\\z&c\end{pmatrix}w = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

8. Consider the matrices

$$A = \begin{pmatrix}2&1&-1\\4&-4&2\end{pmatrix} \qquad B = \begin{pmatrix}5&3&2\\1&-1&2\end{pmatrix} \qquad C = \begin{pmatrix}1&2\\2&-1\end{pmatrix}$$

$$D = \begin{pmatrix}3&1\\-1&2\end{pmatrix} \qquad E = \begin{pmatrix}-3&5\end{pmatrix}.$$

Compute the following expressions. If the result is not defined, explain why.

a) $-3A$ **b)** $B-3A$ **c)** AC **d)** B^2

e) $A+2B$ **f)** $C-E$ **g)** EB **h)** D^2

9. Compute the product

$$\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}2&1&-1\\4&-1&2\end{pmatrix}$$

in two ways:

a) using the column form, and

b) using the outer product form.

10. Consider the matrices

$$A = \begin{pmatrix}1&2\\-2&1\end{pmatrix} \qquad B = \begin{pmatrix}1&1\\-1&h\end{pmatrix}.$$

What value(s) of h, if any, will make $AB = BA$?

11. Show that $(A+B)^2 \neq A^2 + 2AB + B^2$ when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

What is the correct formula?

$$(A+B)^2 = A^2 + B^2 + \underline{\qquad\quad}$$

[**Hint:** distribute the product $(A+B)(A+B)$.]

12. In the following, find the 2×2 matrix A that acts in the specified manner.

a) $A\binom{x}{y} = \binom{x}{y}$: the identity matrix does not change the vector.

b) $A\binom{x}{y} = \binom{y}{x}$: this is a flip over the line $y = x$.

c) $A\binom{x}{y} = \binom{y}{-x}$: this rotates vectors clockwise by 90°.

d) $A\binom{x}{y} = -\binom{x}{y}$: this rotates vectors by 180°.

e) $A\binom{x}{y} = \binom{0}{y}$: this projects onto the y-axis.

f) $A\binom{x}{y} = \binom{x}{0}$: this projects onto the x-axis.

g) $A\binom{x}{y} = \binom{x}{y-2x}$: this performs the row operation $R_2 \mathrel{-}= 2R_1$.

[**Hint:** compute $A\binom{1}{0}$ and $A\binom{0}{1}$.]

13. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by the diagonal matrix D on the right or the left.

14. Let A be a 4×3 matrix satisfying

$$Ae_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 7 \end{pmatrix} \qquad Ae_2 = \begin{pmatrix} 4 \\ 4 \\ -1 \\ -1 \end{pmatrix} \qquad Ae_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

What is A?

15. Suppose that A is a 4×3 matrix such that

$$A\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}7\\1\\2\\9\end{pmatrix} \qquad A\begin{pmatrix}0\\1\\4\end{pmatrix} = \begin{pmatrix}1\\-4\\2\\3\end{pmatrix}.$$

Compute Ax, where x is the vector $(3,-2,-2)$.
[**Hint:** express $(3,-2,-2)$ as a linear combination of $(1,0,2)$ and $(0,1,4)$.]

16. For the following matrices A and B, compute AB, A^T, B^T, B^TA^T, and $(AB)^T$. Which of these matrices are equal and why? Why can't you compute A^TB^T?

$$A = \begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix} \qquad B = \begin{pmatrix}2 & 1 & -1\\4 & -1 & 2\end{pmatrix}.$$

17. Recall that a matrix A is *symmetric* if $A^T = A$. Decide if each statement is true or false, and explain why.

a) If A and B are symmetric of the same size, then AB is symmetric.

b) If A is symmetric, then A^3 is symmetric.

c) If A is any matrix, then A^TA is symmetric.

18. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1\\ -2x_1 + 5x_2 + 5x_3 &= 2\\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Use row operations to eliminate x_1 from all but the first equation.

b) Use row operations to modify the system so that x_2 only appears in the first and second equations (and x_1 still only appears in the first).

c) Solve for x_3, then for x_2, then for x_1. What is the solution?

19. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries of the table.

System of Equations	Matrix Equation	Augmented Matrix
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 \qquad\quad + 4x_3 &= 2 \end{aligned}$		
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	
		$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

20. Which of the following matrices are not in row echelon form? Why not?

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

21. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

$$\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$

22. Find values of a and b such that the following system has **a)** zero, **b)** exactly one, and **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - \ y &= b \end{aligned}$$

23. Give examples of matrices A in *row echelon form* for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?