Math 218D-1: Homework #14

due Wednesday, December 6, at 11:59pm

1. For each matrix A, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

a) $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$

d) $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$ **e)** $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$

2. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 1(a). Let σ_1, σ_2 be the singular values of A. Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

3. Let A be a matrix with nonzero *orthogonal* columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of A in outer product form.

4. Let S be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal eigenbasis, where v_i has eigenvalue λ_i.

a) Show that the singular values of S are $|\lambda_1|, \ldots, |\lambda_r|$. In particular, $\operatorname{rank}(S) = r$.

b) Find the singular value decomposition of S in outer product form, in terms of the λ_i and the v_i.

5. **a)** Show that all singular values of an orthogonal matrix are equal to 1.

b) Let A be an $m \times n$ matrix, let Q_1 be an $m \times m$ orthogonal matrix, and let Q_2 be an $n \times n$ orthogonal matrix. Show that A has the *same singular values* as $Q_1 A Q_2$. [**Hint:** Use HW10#11.]

Remark: This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

6. Let A be a matrix of full column rank and let $A = QR$ be the QR decomposition of A.

a) Show that A and R have the same singular values $\sigma_1, \ldots, \sigma_r$ and the same right singular vectors $v_1, \ldots, v_r$.

b) What is the relationship between the left singular vectors of A and R?

7. Let A be a matrix with first singular value σ_1 and first right singular vector v_1. Recall that the *matrix norm* of A is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$, and is denoted $\|A\|$.

a) Show that $\|Ax\|$ is maximized at $x = v_1$ (subject to $\|x\| = 1$), with maximum value σ_1.

b) Suppose now that A is square and λ is an eigenvalue of A. Show that $|\lambda| \leq \sigma_1$. (You may assume λ is real, although it is also true for complex eigenvalues.)

This shows that *the largest singular value is at least as big as the largest eigenvalue.*

8. **a)** Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 \end{pmatrix}.$$

c) Note that A is very close to A' numerically. Were the eigenvalues of A close to the eigenvalues of A'? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are numerically stable*. This is another advantage of the SVD.

9. Decide if each statement is true or false, and explain why.

a) The left singular vectors of A are eigenvectors of A^TA and the right singular vectors are eigenvectors of AA^T.

b) For any matrix A, the matrices AA^T and A^TA have the same eigenvalues.

c) If S is symmetric, then the nonzero eigenvalues of S are its singular values.

d) If A does not have full column rank, then 0 is a singular value of A.

e) Suppose that A is invertible with singular values $\sigma_1, \ldots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \ldots, \sigma_n + c$.

f) The right singular vectors of A are orthogonal to Nul(A).

10. For each matrix A of Problem 1:

$$\textbf{a)}\ \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

find the singular value decomposition in the matrix form

$$A = U\Sigma V^T.$$

11. For each matrix A of Problem 10, write down orthonormal bases for all four fundamental subspaces. (This can be read off from your answers to Problem 10.)

12. **a)** Let A be an invertible $n \times n$ matrix. Show that the product of the singular values of A equals the absolute value of the product of the (real and complex) eigenvalues of A (counted with algebraic multiplicity).

[**Hint:** Both equal $|\det(A)|$. What is $\det(A^T A)$?]

b) Find an example of a 2×2 matrix A with distinct positive eigenvalues that are not equal to any of the singular values of A.

[**Hint:** One of the matrices in Problem 1 works.]

13. Let S be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $S = QDQ^T$ be an orthogonal diagonalization of S, where D has diagonal entries $\lambda_1, \ldots, \lambda_n$.

Show that $S = QDQ^T$ is a singular value decomposition if and only if S is positive-semidefinite. [See Problem 4.]

14. Let A be a square, invertible matrix with singular values $\sigma_1, \ldots, \sigma_n$.

a) Show that A^{-1} has the same singular vectors as A^T, with singular values $\sigma_n^{-1} \geq \cdots \geq \sigma_1^{-1}$. [**Hint:** What is A^+?]

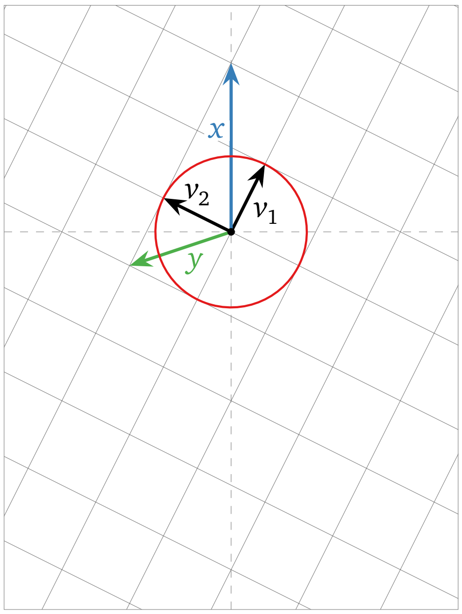
b) Let λ be an eigenvalue of A. Use Problem 7(b) and **a)** to show that $\sigma_n \leq |\lambda|$.

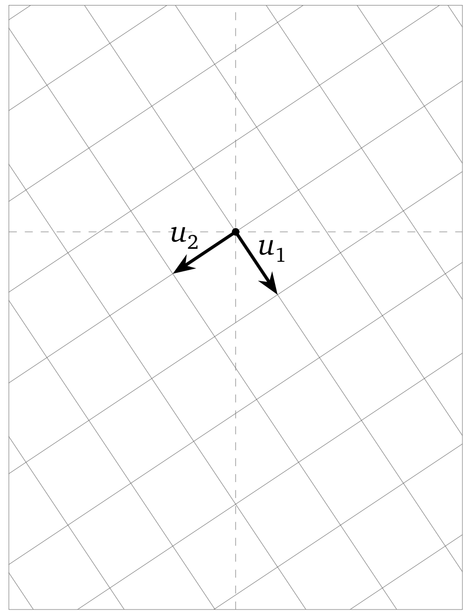
It follows that the absolute values of all eigenvalues of A are contained in the interval $[\sigma_n, \sigma_1]$. Compare Problem 12.

15. A certain 2×2 matrix A has singular values $\sigma_1 = 2$ and $\sigma_2 = 1.5$. The right-singular vectors v_1, v_2 and the left-singular vectors u_1, u_2 are shown in the pictures below.

a) Draw Ax and Ay in the picture on the right.

b) Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit circle by A) in the picture on the right.

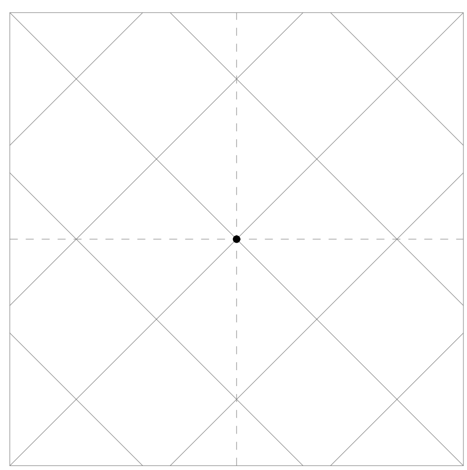




16. Consider the following 3×2 matrix A and its SVD:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^T .$$

Draw $\{Ax : \|x\| = 1\}$ (what you get by multiplying all vectors on the unit sphere by A) in the picture on the right.



17. Compute the pseudoinverse of each matrix of Problem 10.

18. Consider the matrix

$$A = \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

of Problem 17(e). Find the matrix P_V for projection onto $V = \mathrm{Row}(A)$ in two ways:

a) Multiply out $P_V = A^+A$.

b) In Problem 11 you found $\mathrm{Nul}(A) = \mathrm{Span}\{v\}$ for $v = (1, -1, -1, 1)$. Compute $P_{V^\perp} = vv^T/v \cdot v$ and $P_V = I_4 - P_{V^\perp}$.

Your answers to **a)** and **b)** should be the same, of course!

19. Let A be an $m \times n$ matrix.

a) If A has full column rank, show that $A^+A = I_n$.

b) If A has full row rank, show that $AA^+ = I_m$.

In particular, a matrix with full column rank admits a *left inverse*, and a matrix with full row rank admits a *right inverse*. Compare HW5#11.

20. What is the pseudoinverse of the $m \times n$ zero matrix?

21. Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ of Problem 17(b).

a) Find all least-squares solutions of $Ax = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ in parametric vector form.

b) Find the shortest least-squares solution $\widehat{x} = A^+\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

c) Draw your answers to **a)** and **b)** on the grid below.