Math 218D-1: Homework #2

due Wednesday, September 13, at 11:59pm

1. Which of the following matrices are not in reduced row echelon form? Why not?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

2. Describe all possible nonzero 2×2 matrices in RREF.

3. Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use Rabinoff's Reliable Row Reducer.

$$\textbf{a)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \textbf{b)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right) \qquad \textbf{c)} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \qquad \textbf{e)} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array}\right) \qquad \textbf{f)} \left(\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right)$$

4. Determine *how many* solutions each system of equations has. (Do not find the solutions.) [**Hint:** use Problem 3.]

$$\textbf{a)} \begin{cases} x_1 + x_2 \phantom{{}+x_3} = 1 \\ x_1 + x_2 + x_3 = 1 \\ \phantom{x_1 +{}} x_2 + 2x_3 = 2 \end{cases} \qquad \textbf{b)} \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 9x_3 = 1 \end{cases}$$

$$\textbf{c)} \begin{cases} \phantom{3x_1 - {}} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

5. Use Gaussian elimination and back-substitution or Jordan substitution to solve

$$\textbf{a)} \begin{cases} x_1 + x_2 \phantom{{}+2x_3} = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ \phantom{x_1 +{}} x_2 + 2x_3 = 3 \end{cases} \qquad \textbf{b)} \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 8x_3 = 1. \end{cases}$$

What happens if you replace 8 by 9 in (b)?

6. The parabola $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,9)$, $(-1,6)$. Find the coefficients a, b, c.

7. Use the formula for the 2×2 inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

$$\textbf{a)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

8. Compute the inverses of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why.

$$\textbf{a)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 4 \\ -3 & 1 & 4 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 6 & -4 & -7 & -1 \\ 7 & 0 & 1 & 3 \\ -1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

9. Consider the linear system

$$\begin{aligned} x_1 + x_2 \phantom{{}+ 2x_3} &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the Problem 8 to solve for x_1, x_2, x_3 in terms of b_1, b_2, b_3. Do *not* use Gauss–Jordan elimination!

10. Suppose that

$$A\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

What is A^{-1}?

11. Decide if each statement is true or false, and explain why.

a) If A and B are invertible $n \times n$ matrices, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.

b) If A is invertible then so is A^{10}.

c) An $n \times n$ matrix with n pivots is invertible.

d) An invertible $n \times n$ matrix has n pivots.

12. Consider a system of 3 equations in 4 variables. Write the elementary matrices that accomplish the following row operations:

$$\textbf{a)}\ R_2 \mathrel{+}= 2R_1 \qquad \textbf{b)}\ R_1 \mathrel{-}= \tfrac{1}{2}R_3 \qquad \textbf{c)}\ R_3 \mathrel{\times}= 2$$

$$\textbf{d)}\ R_3 \mathrel{\div}= 2 \qquad \textbf{e)}\ R_1 \longleftrightarrow R_3 \qquad \textbf{f)}\ R_1 \longleftrightarrow R_2$$

13. Consider a system of 3 equations in 4 variables. Write the row operations that the following elementary matrices perform on that system:

$$\textbf{a)}\begin{pmatrix}1&3&0\\0&1&0\\0&0&1\end{pmatrix}\qquad \textbf{b)}\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}\qquad \textbf{c)}\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}$$

$$\textbf{d)}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\qquad \textbf{e)}\begin{pmatrix}\frac{1}{4}&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

14. For each elementary matrix in Problem 13, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\ \xrightarrow{\text{row op}}\ R_2 \mathrel{+}= R_1\ \xrightarrow{\text{undo}}\ R_2 \mathrel{-}= R_1\ \xrightarrow{\text{matrix}}\ \begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

15. Consider the matrix

$$A=\begin{pmatrix}1&2&0\\1&3&2\\1&3&1\end{pmatrix}.$$

a) Explain how to reduce A to a matrix U in REF using three row replacements.

b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i:

$$U = \underline{\hspace{3em}}\,A.$$

c) Fill in the blank with a product involving the E_i^{-1}:

$$A = \underline{\hspace{3em}}\,U$$

d) Evaluate that product to produce a lower-triangular matrix L with ones on the diagonal such that $A = LU$.

When multiplying elementary matrices, just use row operations!

e) Explain how to reduce U to the 3×3 identity matrix using three more row operations E_4, E_5, E_6.

f) Fill in the blank with a product involving the E_i:

$$A^{-1} = \underline{\hspace{5em}}.$$

16. Consider the matrix
$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}.$$

a) Perform Gaussian elimination on A without using any row swaps. Write the REF matrix U you obtained.

b) Write the elementary matrices E_1, E_2, E_3 for the row operations you did in (a), with E_1 corresponding to the first row operation.

c) Compute the matrix $L = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$. [**Hint:** Don't multiply matrices! Recall that left-multiplication by E_i^{-1} "un-does" the ith row operation.]

d) Verify that L is lower-unitriangular and that $A = LU$.

17. Solve the following matrix equations by forward- and back-substitution, using the provided LU decomposition. Check your answers by evaluating Ax.

a)
$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

b)
$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

c)
$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

18. Compute the $A = LU$ decomposition of the following matrices using the 2-column method. Check your answers by multiplying LU.

a) $\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$

19. Solve the following matrix equations by forward- and back-substitution, using the provided $PA = LU$ decomposition. Check your answers by evaluating Ax.

a)

$$\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} x = \begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

20. Compute a $PA = LU$ decomposition for each of the following matrices, using the 3-column method and performing *maximal partial pivoting*. Check your answers by multiplying PA and LU.

a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$

21. Recall that a *permutation matrix* is a product of elementary matrices for row swaps.

a) If P is the $n \times n$ elementary matrix for a row swap, explain why $P^{-1} = P = P^T$.

b) If P is any permutation matrix, show that $P^{-1} = P^T$. [**Hint:** write $P = P_1 P_2 \cdots P_r$ for elementary row swaps P_i.] Is $P = P^T$ for a general permutation matrix?