**Math 218D-1: Homework #4**

due Wednesday, September 27, at 11:59pm

**1.** Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

**a)** $\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

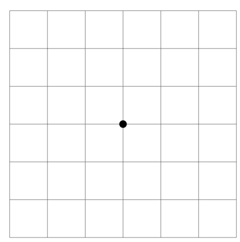
**c)** $\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

[**Hint:** You already did all of the work in HW3#15.]

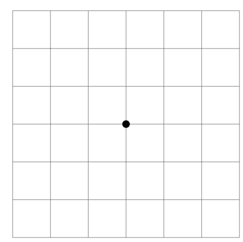
**2.** Draw pictures of the null space and the column space of the following matrices. Be precise!

**a)** $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

Nul($A$)

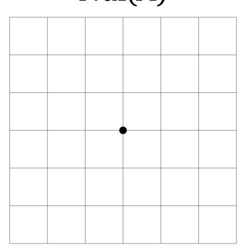


Col($A$)

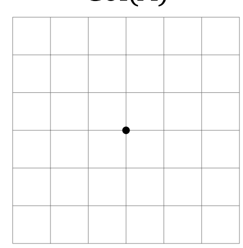


**b)** $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

Nul($A$)

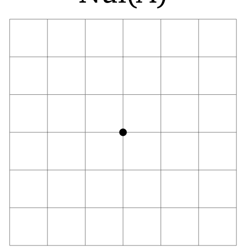


Col($A$)

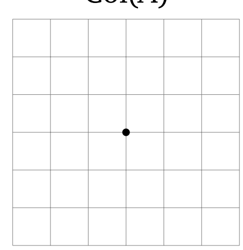


**c)** $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

Nul($A$)



Col($A$)



**3.** Give examples of subsets $V$ of $\mathbf{R}^2$ such that:

**a)** $V$ is closed under addition and contains 0, but is not closed under scalar multiplication.

**b)** $V$ is is closed under scalar multiplication and contains 0, but is not closed under addition.

**c)** $V$ is closed under addition and scalar multiplication, but does not contain 0.

Therefore, none of these conditions is redundant.

**4.** Which of the following subsets of $\mathbf{R}^3$ are subspaces? If it is not a subspace, find a counterexample to one of the subspace properties. If it is, express it as the column space or null space of some matrix.

**a)** The plane $\{(x, y, x) : x, y \in \mathbf{R}\}$.

**b)** The plane $\{(x, y, 1) : x, y \in \mathbf{R}\}$.

**c)** The set consisting of all vectors $(x, y, z)$ such that $xy = 0$.

**d)** The set consisting of all vectors $(x, y, z)$ such that $x \le y$.

**e)** The span of $(1, 2, 3)$ and $(2, 1, -3)$.

**f)** The solution set of the system of equations $\begin{cases} x + \phantom{2}y + z = 0 \\ x - 2y - z = 0. \end{cases}$

**g)** The solution set of the system of equations $\begin{cases} x + \phantom{2}y + z = 0 \\ x - 2y - z = 1. \end{cases}$

**5.** Find a nonzero $2 \times 2$ matrix such that $A^2 = 0$.

**6.** **a)** Explain why $\mathrm{Col}(AB)$ is contained in $\mathrm{Col}(A)$.

**b)** Give an example where $\mathrm{Col}(AB) \ne \mathrm{Col}(A)$.
[**Hint:** Take $A = B$ to be the matrix from Problem 5.]

**7.** **a)** Explain why $\mathrm{Nul}(AB)$ contains $\mathrm{Nul}(B)$.

**b)** Give an example where $\mathrm{Nul}(AB) \ne \mathrm{Nul}(B)$.
[**Hint:** Take $A = B$ to be the matrix from Problem 5.]

**8.** Decide if each statement is true or false, and explain why.

**a)** The column space of an $m \times n$ matrix with $m$ pivots is a subspace of $\mathbf{R}^m$.

**b)** The null space of an $m \times n$ matrix with $n$ pivots is equal to $\mathbf{R}^n$.

**c)** If $\mathrm{Col}(A) = \{0\}$, then $A$ is the zero matrix.

**d)** The column space of $2A$ equals the column space of $A$.

**e)** The null space of $A + B$ contains the null space of $A$.

**f)** If $U$ is an echelon form of $A$, then $\mathrm{Nul}(U) = \mathrm{Nul}(A)$.

**g)** If $U$ is an echelon form of $A$, then $\mathrm{Col}(U) = \mathrm{Col}(A)$.

**9.** **a)** Give an example of a $3 \times 3$ matrix $A$ such that $\mathrm{Col}(A)$ contains $(1,2,3)$ and $(1,0,-1)$, but $\mathrm{Col}(A)$ is not all of $\mathbf{R}^3$. What is the rank of $A$?

**b)** Give an example of a $3 \times 3$ matrix $A$, with no zero entries, such that $\mathrm{Col}(A)$ is the line through $(1,1,1)$. What is the rank of $A$?

**10.** Find a basis for the null space of each matrix.

$$\textbf{a)}\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

$$\textbf{c)}\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad \textbf{d)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[**Hint:** You already did all of the work in Problem 1.]

**11.** Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear relation among them.

$$\textbf{a)}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\} \qquad \textbf{b)}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\} \qquad \textbf{c)}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$

$$\textbf{d)}\left\{\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix}, \begin{pmatrix}2\\-4\\2\\-6\end{pmatrix}, \begin{pmatrix}3\\-5\\2\\-7\end{pmatrix}, \begin{pmatrix}-1\\4\\-3\\7\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\6\end{pmatrix}\right\} \qquad \textbf{e)}\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$$

Which sets do you know are linearly dependent without doing any work?

**12.** **a)** For each set in Problem 11, find a basis for the span of the vectors.

**b)** For each set in Problem 11, find a *different* basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for **a)**.

**c)** What is the dimension of each of these spans?

**13.** Consider the vectors

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$$

of Problem 11(a).

**a)** Find two different ways to express $(5,7,9)$ as a linear combination of these vectors.

**b)** How many ways can you express $(5,7,9)$ as a linear combination of the first two vectors?

**14.** Let $\{w_1, w_2, w_3\}$ be a basis for a subspace $V$, and set

$$v_1 = w_2 + w_3 \qquad v_2 = w_1 + w_3 \qquad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for $V$.

[**Hint:** You have to check spanning and linear independence by hand.]

**15.** Certain vectors $v_1, v_2, v_3, v_4$ span a 3-dimensional subspace of $\mathbf{R}^5$. They satisfy the linear relation

$$2v_1 + 0v_2 - v_3 + v_4 = 0.$$

**a)** Describe *all* linear relations among $v_1, v_2, v_3, v_4$.
[**Hint:** what is the rank of the matrix with columns $v_1, v_2, v_3, v_4$?]

**b)** Which vector(s) is/are *not* in the span of the others? How do you know for sure?

**16.** Consider the following matrix:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

Which sets of columns form a basis for the column space? (I.e., do the first and third columns form a basis? what about the second and third? etc.)

**17.** Find bases for the following subspaces.

**a)** $\{(x, y, x) : x, y \in \mathbf{R}\}$.

**b)** $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

**c)** The solution set of the system of equations $\begin{cases} x + \phantom{2}y + z = 0 \\ x - 2y - z = 0. \end{cases}$

**d)** $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

**e)** The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

**f)** The intersection of the plane $x - 2y - z = 0$ with the $xy$-plane.

**18.** Let $V$ be a 4-dimensional subspace of $\mathbf{R}^5$.

**a)** Explain why every basis for $V$ can be extended to a basis for $\mathbf{R}^5$ by adding one more vector.

**b)** Find an example of a 4-dimensional subspace $V$ of $\mathbf{R}^5$ and a basis for $\mathbf{R}^5$ that cannot be reduced to a basis for $V$ by removing one vector.

**19.** Decide if each statement is true or false, and explain why.

**a)** If $v_1, v_2, \ldots, v_n$ are linearly independent vectors, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ has dimension $n$.

**b)** If the matrix equation $Ax = 0$ has the trivial solution, then the columns of $A$ are linearly independent.

**c)** If $\text{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \text{Span}\{v_1, v_2\}$.

**d)** If $v_3$ is not a linear combination of $v_1$ and $v_2$, then $\{v_1, v_2, v_3\}$ is linearly independent.

**e)** If $\{v_1, v_2, v_3\}$ is linearly dependent, then so is $\{v_1, v_2, v_3, x\}$ for any vector $x$.

**f)** The set $\{0\}$ is linearly independent.

**g)** If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then so is $\{v_1, v_2, v_3\}$.

**h)** The columns of any $4 \times 5$ matrix are linearly dependent.

**i)** If $Ax = \binom{1}{2}$ has only one solution, then the columns of $A$ are linearly independent.

**j)** If $\text{Span}\{v_1, v_2, v_3\}$ has dimension 3, then $\{v_1, v_2, v_3\}$ is linearly independent.