Math 218D-1: Homework #6

due Wednesday, October 11, at 11:59pm

1. For each pair of vectors v and b, draw Span$\{v\}$, and compute and draw the projection b_V of b onto $V = \text{Span}\{v\}$.

$$\textbf{a) } v = \begin{pmatrix}1\\0\end{pmatrix},\ b = \begin{pmatrix}\cos(123^\circ)\\ \sin(123^\circ)\end{pmatrix} \qquad \textbf{b) } v = \begin{pmatrix}-1\\1\end{pmatrix},\ b = \begin{pmatrix}1\\1\end{pmatrix}$$

2. For each subspace V and vector b, compute the orthogonal projection b_V of b onto V by solving a normal equation $A^TAx = A^Tb$, and find the distance from b to V.

a) $$V = \text{Col}\begin{pmatrix}1 & 1\\ 1 & 0\\ 0 & 2\end{pmatrix} \qquad b = \begin{pmatrix}1\\4\\3\end{pmatrix}$$

b) $$V = \text{Col}\begin{pmatrix}1 & 2 & 1\\ -1 & 1 & 0\\ 2 & 2 & -1\\ 4 & 3 & 0\end{pmatrix} \qquad b = \begin{pmatrix}-1\\-1\\-1\\7\end{pmatrix}$$

c) $$V = \text{Col}\begin{pmatrix}2 & 2 & -1\\ -4 & -5 & 5\\ 6 & 1 & 12\end{pmatrix} \qquad b = \begin{pmatrix}-6\\-24\\-3\end{pmatrix}$$

3. For each subspace V, compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1, 2, -1)$ with respect to V.

$$\textbf{a) } V = \text{Span}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}\right\} \qquad \textbf{b) } V = \text{Nul}\begin{pmatrix}1 & 2 & 2\\ 0 & 2 & 0\end{pmatrix}$$

$$\textbf{c) } V = \mathbf{R}^3 \qquad \textbf{d) } V = \{0\}$$

[**Hint:** Only part **a)** requires any work.]

4. Compute the orthogonal decomposition $(3, 1, 3) = b_V + b_{V^\perp}$ with respect to each subspace of V of HW5#18(a)–(e).

[**Hint:** Only parts **a)** and **c)** require any work, and even **c)** doesn't require work if you're clever enough. In fact, you can solve all five parts by computing two dot products.]

5. **a)** Let $v, w \in \mathbf{R}^n$. Show that

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2$$

if $v \perp w$.

b) Let V be a subspace of $\mathbf{R}^n$, let $b \in \mathbf{R}^n$, and let $v \in V$. Use **a)** and the fact that $b - b_V \in V^\perp$ to show that

$$\|b-v\|^2 = \|b-b_V\|^2 + \|b_V - v\|^2.$$

Use this to prove that b_V really is the closest vector in V to b.

c) Let V be a subspace of $\mathbf{R}^n$ and let $b \in \mathbf{R}^n$. Use **a)** to show that $\|b_V\| \leq \|b\|$, with equality if and only if $b \in V$.

6. Let A be an $m \times n$ matrix, and let $b \in \mathbf{R}^m$ be a vector. Suppose that $A^T b = 0$. Compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ with respect to $V = \mathrm{Col}(A)$.

7. **a)** Find an implicit equation for the plane

$$\mathrm{Span}\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix} \right\}.$$

[**Hint:** use HW5#18(a).]

b) Find implicit equations for the line $\{(t, -t, t) : t \in \mathbf{R}\}$.
[**Hint:** use HW5#19(g).]

8. Show that $A^T A = 0$ is only possible when $A = 0$.

9. Let Q be an $n \times n$ (*square*) matrix such that $Q^T Q = I_n$ (so $Q^T = Q^{-1}$).

a) Show that the columns of Q are unit vectors.

b) Show that the columns of Q are orthogonal to each other.

c) Show that the *rows* of Q are also orthogonal unit vectors.

d) Find all 2×2 matrices Q such that $Q^T Q = I_2$.

Such a matrix Q is called *orthogonal*.[1]

10. Explain why A has full column rank if and only if $A^T A$ is invertible.

[1] I am not responsible for this terminology.

11. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal complements.

b) If A is a 3×4 matrix, then $\mathrm{Col}(A)^\perp$ is a subspace of $\mathbf{R}^4$.

c) If A is any matrix, then $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A)$.

d) If A is any matrix, then $\mathrm{Row}(A) = \mathrm{Row}(A^T A)$.

e) If every vector in a subspace V is orthogonal to every vector in another subspace W, then $V = W^\perp$.

f) If $x \in V$ and $x \in V^\perp$, then $x = 0$.

g) If x is in a subspace V, then the orthogonal projection of x onto V is x.

h) If x is in the orthogonal complement of a subspace V, then the orthogonal projection of x onto V is x.

12. For each column space V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

a) $V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$ **b)** $V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix}$ **c)** $V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

13. For each subspace V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

a) $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\}$ **b)** $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

14. For each vector v, compute the projection matrix onto $V = \mathrm{Span}\{v\}$ using the formula $P_V = vv^T / v \cdot v$.

a) $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ **b)** $v = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix}$ **c)** $v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ (in $\mathbf{R}^n$)

15. **a)** Compute P_V for $V = \mathbf{R}^n$.

b) Compute P_V for $V = \{0\}$.

16. For each subspace V, compute the projection matrix P_V.

a) $\{(x,y,x) : x,y \in \mathbf{R}\}$.

b) $\{(x,y,z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

g) The line $\{(t, -t, t) : t \in \mathbf{R}\}$.

[**Hint:** Compare HW4#17 and HW5#19. You can save a lot of work by sometimes computing $P_{V^\perp}$ and using $P_V = I_3 - P_{V^\perp}$.]

17. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 3 \end{pmatrix},$$

and let $V = \mathrm{Col}(A)$.

a) Compute P_V using the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute a basis $\{v_1, v_2\}$ for $V^\perp = \mathrm{Nul}(A^T)$.

c) Let B be the matrix with columns v_1, v_2, and compute $P_{V^\perp}$ using the formula $B(B^TB)^{-1}B^T$.

d) Verify that your answers to (a) and (c) sum to I_4.

(Factor out $ad - bc$ and use a computer to do the matrix multiplication! Your answers should be in fractions, not decimals.)

This illustrates the fact that once you've computed P_V, there's no need to compute $P_{V^\perp}$ separately. It's a lot of extra work!

18. Compute the matrices P_1, P_2 for orthogonal projection onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$, respectively. Now compute P_1P_2, and explain why it is what it is.

19. Consider the plane V defined by the equation $x+2y-z=0$. Compute the matrix P_V for orthogonal projection onto V in two ways:

a) Find a basis for V, put your basis vectors into a matrix A, and use the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute the matrix for orthogonal projection $P_{V^\perp}$ onto the line $V^\perp$ using the formula $vv^T/v\cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.
[**Hint:** It doesn't take any work to find a basis for $V^\perp$.]

If V is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

20. Decide if each statement is true or false, and explain why. In each statement, V is a subspace of $\mathbf{R}^n$.

a) The rank of P_V is equal to $\dim(V)$.

b) $P_V P_{V^\perp} = 0$.

c) $P_V + P_{V^\perp} = 0$.

d) $\mathrm{Col}(P_V) = V$.

e) $\mathrm{Nul}(P_V) = V$.

f) $\mathrm{Row}(P_V) = \mathrm{Col}(P_V)$.

g) $\mathrm{Nul}(P_V)^\perp = \mathrm{Col}(P_V)$.