Math 218D-1: Homework #8

due Wednesday, October 25, at 11:59pm

1. Compute the determinants of the following matrices *using Gaussian elimination*.

a) $\begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix}$

c) $\begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix}$ **d)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

2. Suppose that

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10.$$

Find the determinants of the following matrices.

a) $\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$ **b)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g+2d & h+2e & i+2f \end{pmatrix}$ **c)** $\begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix}$

d) $\begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$ **e)** $\begin{pmatrix} a & b & c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix}$ **f)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

g) $2\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **h)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **i)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}$

j) $-\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **k)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{3}$ **l)** $\begin{pmatrix} a & b+2c & c \\ d & e+2f & f \\ g & h+2i & i \end{pmatrix}$

3. Find $\det(E)$ when:

a) E is the elementary matrix for a row replacement.

b) E is the elementary matrix for $R_i \times\!= c$.

c) E is the elementary matrix for a row swap.

4. A matrix A has the $PA = LU$ factorization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A = L \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

What is $\det(A)$?.

5. Let A be the $n \times n$ matrix $(n \geq 3)$ whose (i, j) entry is $i + j$. Compute $\det(A)$.

6. Recall that an *orthogonal matrix* is a square matrix with orthonormal columns, or equivalently, a square matrix Q such that $Q^T Q = I_n$. Prove that every orthogonal matrix has determinant ± 1.

7. Let V be a subspace of $\mathbf{R}^n$ and let P_V be the projection matrix onto V.

a) Find $\det(P_V)$ when $V \neq \mathbf{R}^n$.

b) Find $\det(P_V)$ when $V = \mathbf{R}^n$.

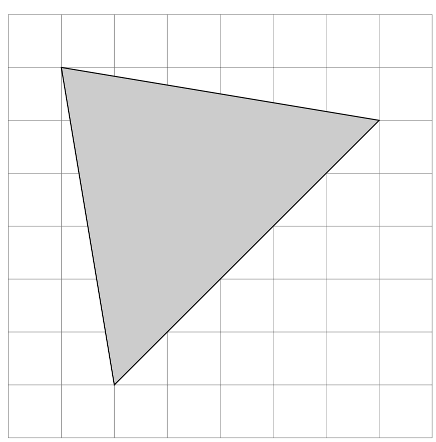
8. Let C be the *hypercube* in $\mathbf{R}^4$ with corners $(\pm 1, \pm 1, \pm 1, \pm 1)$. Compute the volume of C.

9. Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$.

a) Show that if $\{v_1, v_2, \ldots, v_n\}$ is orthogonal then $|\det(A)| = \|v_1\| \, \|v_2\| \cdots \|v_n\|$.
[**Hint:** Compute $A^T A$ and its determinant.]

b) Suppose that A is invertible. Show that $|\det(A)| \leq \|v_1\| \, \|v_2\| \cdots \|v_n\|$, with equality if and only if the set $\{v_1, v_2, \ldots, v_n\}$ is orthogonal.
[**Hint:** Use HW6#5(c) and the QR decomposition of A.]

10. Compute the area of the triangle pictured below using a 2×2 determinant. (The grid marks are one unit apart.)



11. Decide if each statement is true or false, and explain why.

a) $\det(A+B) = \det(A) + \det(B)$.

b) $\det(ABC^{-1}) = \dfrac{\det(A)\det(B)}{\det(C)}$.

c) $\det(AB) = \det(BA)$.

d) $\det(3A) = 3\det(A)$.

e) If A^5 is invertible then A is invertible.

f) The determinant of A is the product of its diagonal entries.

g) If the columns of A are linearly dependent, then $\det(A) = 0$.

h) If A is a 3×3 matrix with determinant zero, then two of the columns of A are scalar multiples of each other.