

Welcome to Math 218D-1!

Introduction to Linear Algebra

What is Linear Algebra?

The study of (systems of) linear equations

Like: $y = 3x + 2 \rightsquigarrow -3x + y = 2$
(usually put variables on the left & constants on the right)

Or: $\begin{cases} x + y + z = 1 \\ y - z = -3 \end{cases}$; solve both equations at once
(arrange in columns to keep things tidy)

Linear means: equations that involve only sums of (number) · (variable) or (number)

Not: $xy + z = 1$
↑ product of variables

$x + 3 = y^2$
power of a variable

$e^x = \cos(y)$
↑ complicated functions

Why linear algebra?

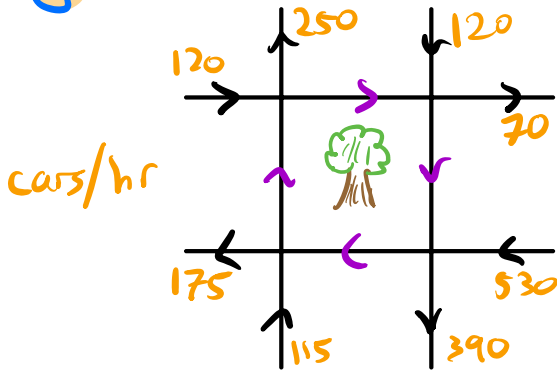
- It's simple enough to understand very well & program computers to do quickly.

- It's powerful enough to solve a huge range of different problems.

"example"

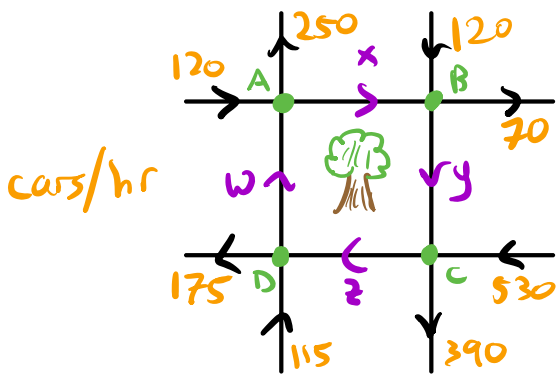
Eg:

Here's a map of roads in the town square:



Question: How many cars/hr travel on the unlabeled roads?

Step 0: When you have an unknown quantity, give it a name!



Observation: #cars entering each intersection = #cars leaving it

(notice nice columns)

$$\begin{aligned}
 A: & 120 + w = 250 + x \\
 B: & 120 + x = 70 + y \\
 C: & 530 + y = 390 + z \\
 D: & 115 + z = 175 + w
 \end{aligned}$$

$$\begin{cases}
 -x & & & +w & = 130 \\
 x & -y & & & = -50 \\
 & y & -z & & = -140 \\
 & & z & -w & = 60
 \end{cases}$$

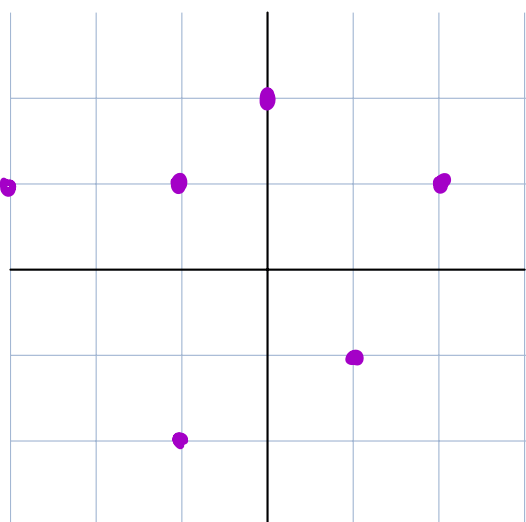
This is a system of 4 linear equations in 4 unknowns!

Question: You know a priori that there are infinitely many solutions. How?

Question: What must be true about the known quantities for a solution to exist?

Linear algebra is a set of tools for solving equations. It is your job to turn your question into a linear algebra problem (that a computer can solve) and interpret the answer.

Eg: An asteroid has been observed at coordinates: $(0,2)$, $(2,1)$, $(1,-1)$, $(-1,-2)$, $(-3,1)$, $(-1,1)$



Question: What is the most likely orbit? Will the asteroid crash into the Earth?

Fact: The orbit is an ellipse.

Equation for an ellipse:

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

Wait! Isn't this a nonlinear equation? ...

For our points to lie on the ellipse, substitute the coordinates into $(x, y) \rightsquigarrow$ these should hold:

$$\begin{array}{l} \overset{x}{=} \overset{y}{=} \\ (0, 2): \quad 0 + 4B + 0 + 0 + 2E + F = 0 \\ (2, 1): \quad 4 + B + 2C + 2D + E + F = 0 \\ (1, -1): \quad 1 + B - C + D - E + F = 0 \\ (-1, -2): \quad 1 + 4B + 2C - D - 2E + F = 0 \\ (-3, 1): \quad 9 + B - 3C + D - 3E + F = 0 \\ (-1, 1): \quad 1 + B - C - D + E + F = 0 \end{array}$$

This is a system of six linear equations in 5 variables.

Note \rightarrow **NB:** The variables are the coefficients B, C, D, E, F .

Remember, we're finding the equation of the ellipse.

NB: There is no solution — the points do not lie on an ellipse (perhaps due to measurement error).

Question: What is the best approximate solution?

\rightsquigarrow "least squares" (week 8)

Answer: [demo]

Historical note: Gauss invented much of what you'll learn to (correctly) predict the orbit of the asteroid Ceres in 1801.

Note on demos: I created these to help give you a **geometric** understanding of linear algebra.

- It took a lot of work.
- Really, it was hard.
- Why would I do that? I want you to have a geometric understanding.

Upshot: Play with the demos! Don't turn off your brain when we do geometry! You will be expected to draw pictures on exams!

Eg: In a population of rabbits,

- (1) Half survive their first year 😞
- (2) Half of those survive their second year.
- (3) The maximum life span is 3 years.
- (4) Each rabbit produces (on average) 0, 6, 8 offspring in years 0, 1, 2, respectively.

Question: How many rabbits will there be in 100 years?

Step 0: Give names to the unknowns.

X_n : # rabbits aged 0 in year n

Y_n : # rabbits aged 1 in year n

Z_n : # rabbits aged 2 in year n

Rules: $X_{2021} = 6Y_{2020} + 8Z_{2020}$

$$Y_{2021} = \frac{1}{2}X_{2020}$$

$$Z_{2021} = \frac{1}{2}Y_{2020}$$

A system of equations of this form is called a **difference equation**. We'll solve them using **eigenvalues & diagonalization** (week 10).

[demo] It looks like eventually,

- The population doubles each year
- The ratio of rabbits aged 0:1:2 is $\approx 16:4:1$

Comes from: $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{bmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$ w/ eigenvalue 2.

Other examples:

- Google PageRank lets you search the Web with a Markov chain — a special type of difference equation.

- Netflix knows what movies you'll like using the Singular Value Decomposition (weeks 14-15).

Geometry of Solutions

Convention: given a system of linear equations, put the **constant** term on the **right** of the $=$, and put the **variables** on the **left**, organized in columns.

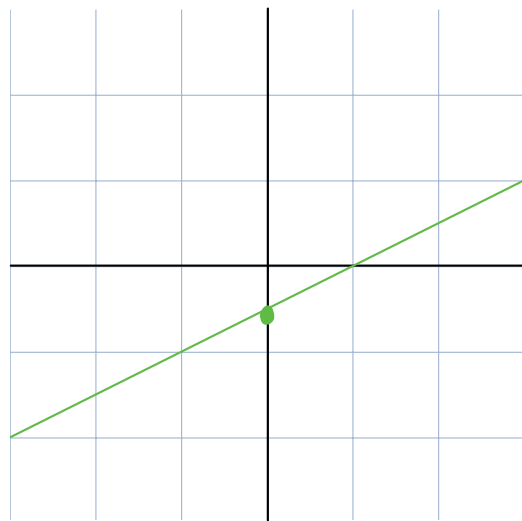
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 \end{array}$$

Def: The **solution set** of a system of equations is the set of all values for the variables making all equations true **simultaneously**.

Question: What does the solution set of a system of **linear** equations look like?

One equation in 2 variables:

$$x - 2y = 1 \quad \rightsquigarrow \quad y = \frac{1}{2}x - \frac{1}{2}$$



One equation in 3 variables:

$$x + y + z = 1 \quad \rightsquigarrow \quad z = 1 - x - y$$

plane in xyz -space
[demo]

One equation in 4 variables: "3-plane in 4-space"

Note on dimensions: Students often want to say "the fourth dimension is time". Einstein used \mathbb{R}^4 (4-space) to model spacetime, but it models lots of other things too. (like traffic around the town square...)

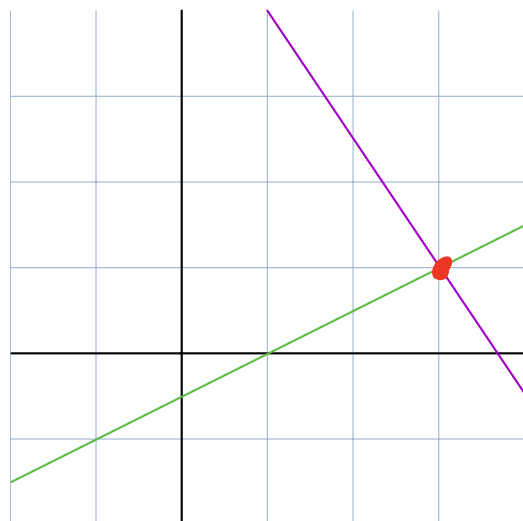
2 equations in 2 variables:

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

Where are both true?

Intersection of 2 lines.

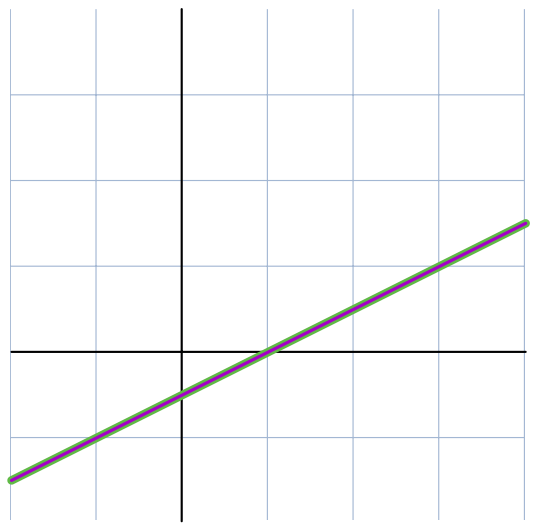
(answer: $(3, 1)$)



What else can happen?

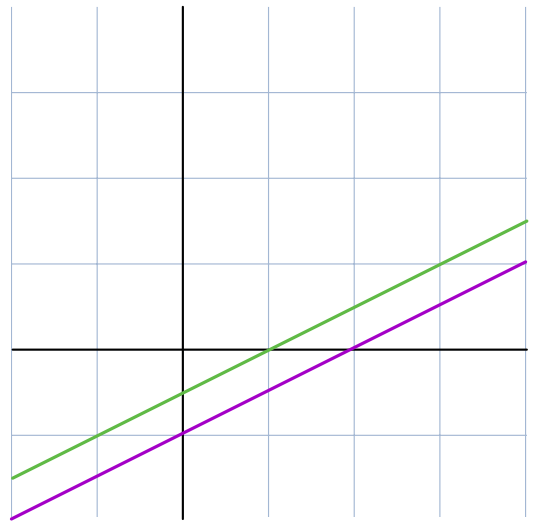
$$\begin{aligned}x - 2y &= 1 \\ 3x - 6y &= 3\end{aligned}$$

Same line: ∞ solutions.



$$\begin{aligned}x - 2y &= 1 \\ 3x - 6y &= 6\end{aligned}$$

Parallel lines: 0 solutions



2 equations in 3 variables:

$$\begin{aligned}x + y + z &= 1 \\ x - z &= 0\end{aligned} \quad \rightsquigarrow \quad \begin{array}{l} \text{intersection of two planes} \\ \text{in space [demo]} \end{array}$$

In this case, it's a line.

3 equations in 3 variables:

$$\begin{aligned}x + y + z &= 1 \\ x - z &= 0 \\ y &= 0\end{aligned} \quad \rightarrow \quad \begin{array}{l} x = 1/2 \\ y = 0 \\ z = 1/2 \end{array}$$

intersection of three planes in space: in this case it's a point.

Question: How many "ways" can 3 planes in space intersect?

Answer: 8

Syllabus Stuff: see the syllabus for details.

- Course materials, calendar, resources, links, etc. are on the **course webpage**:

<https://services.math.duke.edu/~ujdr/2324f-218/>

- We will use **Sakai** for:

→ Announcements

→ Gradebook

→ Gradescope

!! Please use the **Gradescope** tab on Sakai instead of going to [gradescope.com](https://www.gradescope.com).

→ **Ed Discussion**: for asking questions (replaces Piazza).

!! Don't email us w/math questions! Post it here instead - then everyone sees it & anyone can answer.

→ **WarpWire** (see below)

Textbook:

- Strang, "Introduction to Linear Algebra", 5th ed. We'll only follow this loosely. Also see
- Margalit-Rabinoff, "Interactive Linear Algebra" (on the course website). You'll get a link to a **beta version** aimed just at this course!

Quizzes: a 10-minute small-group quiz will be held at the beginning of each discussion section. It's very basic - just tests if you've looked over your notes.

Homework: due Wednesday 11:59pm every week.

- Meant to be long and hard: you need **practice** to learn math, and practice takes **time**.
- Scan & submit on Gradescope.
Use a scanning app!
- **Tag** the pages on Gradescope with the problems on that page!

Midterms: 2 of them, during discussion slots.

Final: as scheduled by the registrar.

Help! • Come to office hours!

- Ask on Ed Discussion
- See course webpage.

Recorded Lecture:

Basics of vector & matrix algebra.

Watch before Thursday (on Warp Wire)

HW#1 covers that material.