Determinants

What we've done:

- Solve $Ax=b$

  (Gauss-Jordan, LU, PVF, ...)

- Approximately solve $Ax=b$

  (orthogonality, projections, QR, ...)

What's next:

- Solve $Ax=\lambda x$

This is the eigenvalue problem used in difference equations (rabbit population) & ODEs. It deals exclusively with square matrices.

The determinant of a square matrix is a number that satisfies many magical properties. I'll define it by telling you how to compute it using row operations.

→ Next time: other ways to compute it.

Def: The determinant of a square matrix A is a number $\det(A)$ or $|A|$ satisfying:

(1) If $A \xrightarrow{R_i += cR_j} B$ then $\det(A) = \det(B)$.

(2) If $A \xrightarrow{R_i \times = c} B$ then $\det(A) = \frac{1}{c}\det(B)$.

(3) If $A \xrightarrow{R_i \leftrightarrow R_j} B$ then $\det(A) = -\det(B)$

(4) $\det(I_n) = 1$.

Consequence: if A has a zero row then $\det(A) = 0$

Eg: $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[(2)]{R_3 \times = -1} -\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Longrightarrow \det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Consequence: if A is (upper/lower) triangular then $\det(A)$ = product of diagonal entries

$$\boxed{\det\begin{pmatrix} \text{triangular} \\ \text{matrix} \end{pmatrix} = \text{product of the diagonal entries}}$$

eg. REF

Eg: $\det\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[R_3 \times = \frac{1}{c}]{R_1 \times = \frac{1}{a} \ R_2 \times = \frac{1}{b}} abc \det\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ (2)

$\xrightarrow[\text{replacements}]{\text{row (1)}} abc \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{(4)}{=} abc$

What if $b=0$ though? (or a? or c?)

$\det\begin{pmatrix} a & * & * \\ 0 & 0 & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\text{replacements}]{\text{row (1)}} \det\begin{pmatrix} a & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$

$= 0 = a \cdot 0 \cdot c$ ✓

A REF matrix is triangular, so you can compute $\det(A)$ by Gaussian elimination!

Eg: $\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow[(3)]{R_1 \leftrightarrow R_2} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$

$\xrightarrow[(1)]{R_3 -= R_1} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[(1)]{R_3 += R_2} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

$= -2$

NB: You get the same number for $\det(A)$ nomatter which row operations you do!

Eg: $\det\begin{pmatrix}0&1&3\\1&2&1\\1&1&0\end{pmatrix} \underset{(3)}{\overset{R_1\leftrightarrow R_3}{=}} -\det\begin{pmatrix}1&1&0\\1&2&1\\0&1&3\end{pmatrix}$

$\underset{(1)}{\overset{R_2 -= R_1}{=}} -\det\begin{pmatrix}1&1&0\\0&1&1\\0&1&3\end{pmatrix} \underset{(1)}{\overset{R_3 -= R_2}{=}} -\det\begin{pmatrix}1&1&0\\0&1&1\\0&0&2\end{pmatrix}$

$= -2$ ✓

Gaussian elimination is the fastest general algorithm for computing the determinant of a matrix (with known entries).

Procedure: To compute $\det(A)$, run Gaussian elimination: $A \xrightarrow[\text{operations}]{\text{row}} U$. Then

$$\boxed{\det(A) = (-1)^{\#\text{ row swaps}} \cdot \frac{1}{\prod(\text{row scaling})} \prod\begin{pmatrix}\text{diagonal}\\ \text{entries of } U\end{pmatrix}}$$

product

NB: You don't need to do row scaling operations to run Gaussian elimination, so this term usually does not appear.

NB: Row operations multiply det by a nonzero scalar:

$A \xrightarrow[\text{ops}]{\text{row}} B \implies \det(B) = \begin{pmatrix}\text{nonzero}\\ \text{number}\end{pmatrix} \cdot \det(A)$.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- If $a \neq 0$:

$$\det(A) \xlongequal{R_2 -= \frac{c}{a}R_1} \det\begin{pmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{pmatrix}$$

$$= a\left(d - \frac{c}{a}b\right) = ad - bc$$

- If $a = 0$:

$$\det(A) \xlongequal{R_1 \leftrightarrow R_2} -\det\begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

$$= -bc = ad - bc$$

$$\boxed{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc}$$

Magical Properties of the Determinant:

(1) Existence: There exists a number $\det(A)$ satisfying defining properties (1)–(4).

(2) Invertibility: A is invertible $\Leftrightarrow \det(A) \neq 0$

(3) Multiplicativity: $\det(AB) = \det(A)\det(B)$

and $\det(A) \neq 0 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

(4) Transposes: $\det(A^T) = \det(A)$

We'll only prove (2) in class. See ILA for the rest.

Existence

(1) says: You get the same number for det(A) nomatter which row ops you do!

Invertibility

Proof: If U is a REF of A then

$\det(U) =$ product of diagonal entries

$\det(U) \neq 0 \iff$ all diagonal entries are nonzero

$\iff A$ has n pivots

$\iff A$ is invertible

We know $\det(A) = (\text{nonzero scalar}) \cdot \det(U)$,

so also $\det(A) \neq 0 \iff \det(U) \neq 0$. ✓

Eg: $\det\begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix} = (-1)(-3)-(1)(3)=0$

so $\begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix}$ is singular (not invertible)

NB: If the columns of A are linearly dependent then A does not have full column rank $\Rightarrow$ not invertible $\Rightarrow \det(A)=0$. Likewise for rows (take transposes).

$$\boxed{A \text{ has linearly dependent rows or columns} \Rightarrow \det(A)=0}$$

Multiplicativity

Eg: $\det\left[\begin{pmatrix}0 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}^{100}\right]$

$$= \det\left[\begin{pmatrix}0 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}^{99}\right]$$

$$= \det\begin{pmatrix}0 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}\det\left[\begin{pmatrix}0 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}^{98}\right]$$

$$= \cdots = \left[\det\begin{pmatrix}0 & 1 & 3\\ 1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}\right]^{100} = (-2)^{100}$$

More generally,

$$\boxed{\det(A^n) = \det(A)^n \quad \text{for all } n \geq 0 \text{ (and } n<0 \text{ if } \det(A) \neq 0)}$$

Eg: Say A has a $PA = LU$ decomposition.

$$\det(L) = \det\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{pmatrix} = 1$$

You get P by doing row swaps on I_n, so

$$\det(P) = (-1)^{\#\text{row swaps}}$$

Hence

$$\begin{aligned} (-1)^{\#\text{row swaps}} \det(A) &= \det(PA) \\ &= \det(LU) = \det(L)\det(U) \\ &= \det(U) \end{aligned}$$

This recovers the formula on p.4 (we did no row scaling operations).

Transposes

The transpose property says that $\det(A)$ satisfies (1)–(3) for column operations too: they're just row operations on A^T.

Eg:

$$\det\begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \xlongequal{C_1 -= 4C_3} \det\begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Big\| \text{transpose} \qquad\qquad\qquad \Big\| \text{transpose}$$

$$\det\begin{pmatrix} 2 & 3 & 4 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} \xlongequal{R_1 -= 4R_3} \det\begin{pmatrix} -14 & -9 & 0 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$$

So we can compute det using column ops:

$$\det\begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \xlongequal{C_1 -= 4C_3} \det\begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

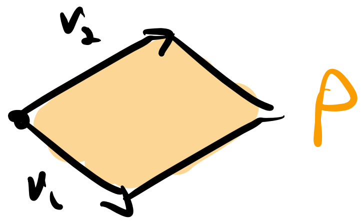
$$\xlongequal{C_1 += 9C_2} \det\begin{pmatrix} 49 & 7 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 49$$

Determinants and Volumes

Where do properties (1)–(4) come from?

Two vectors $v_1, v_2 \in \mathbb{R}^2$ determine ("span") a parallelogram:



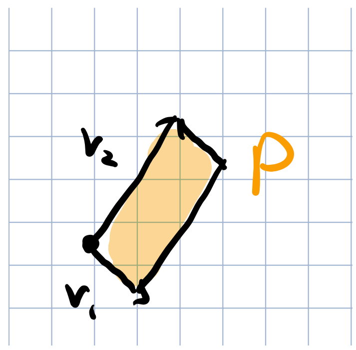
$$P = \{x_1 v_1 + x_2 v_2 : x_1, x_2 \in [0,1]\}$$

Fact: $\text{area}(P) = \left|\det\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix}\right|$

Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\text{area}(P) = \left|\det\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\right|$$

$$= |(3)(1) - (2)(-1)| = 5$$

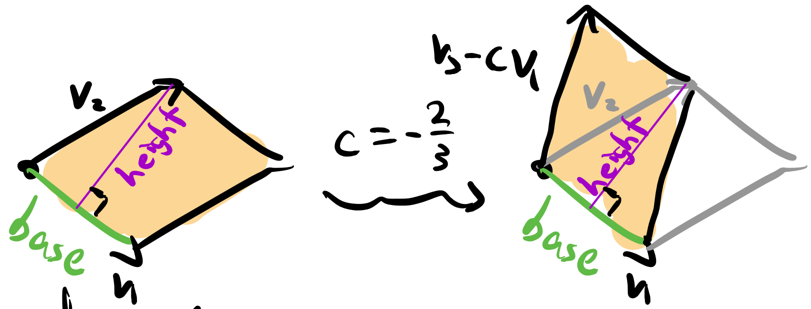


Why? Let's check that area(P) satisfies the four defining properties (1)–(4) of the determinant.

NB: area(P) = base × height:

(1) Row replacement

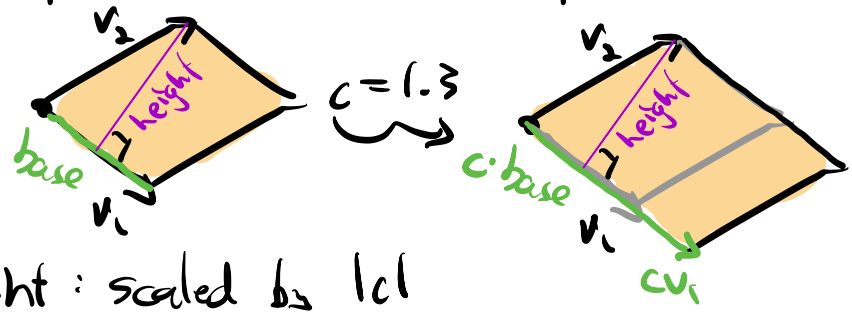
$v_2 \rightsquigarrow v_2 + cv_1$



area = base × ht : unchanged

(2) Row scaling

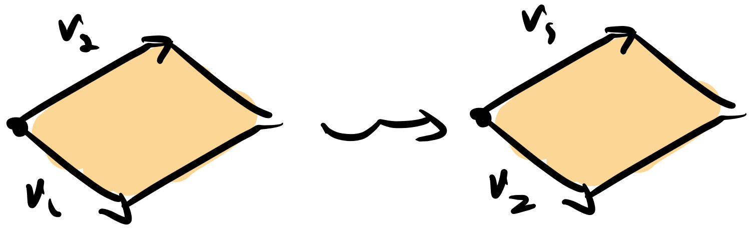
$v_1 \rightsquigarrow cv_1$



base scaled by $|c| \Rightarrow$ base × ht : scaled by $|c|$

(3) Row Swap

$v_1 \longleftrightarrow v_2$



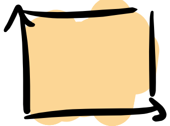
area unchanged = |det|

Q: minus sign?

HW 9

(4) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

area = 1.

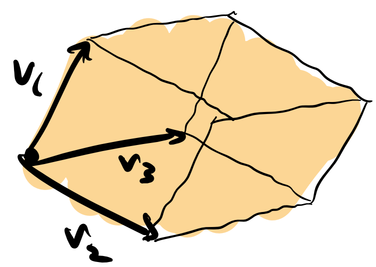


This generalizes as follows (same reasoning):

Def: The paralellepiped determined ("spanned") by n vectors $v_1, \dots, v_n \in \mathbb{R}^n$ is

$$P = \{x_1 v_1 + \cdots + x_n v_n : x_1, \dots, x_n \in [0,1]\}$$

P



Thm (Determinants & Volumes):

$$\text{volume}(P) = \left|\det\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}\right|$$

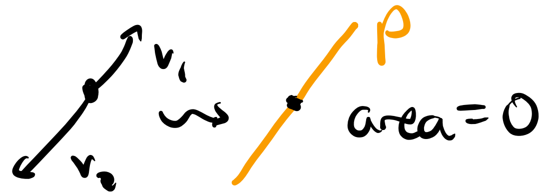
NB: When $n=1$, "volume" = "length":

$$\text{length}(a) = |a|$$

NB: When $n=2$, "volume" = "area".

Question: When is $\text{volume}(P)=0$?

When P is squashed flat:



ie when $v_1, \dots, v_n$ are linearly dependent ($\Rightarrow \det(\cdots)=0$)

NB: In multivariable calc, you approximate shapes by tiny cubes, which turn into tiny parallelepipeds after applying a function. This is why determinants appear in the change of variables formula for integrals:

if $(y_1, \dots, y_n) = f(x_1, \dots, x_n)$ then

$$dy_1 \cdots dy_n = \det\left(\partial y_i / \partial x_j\right) dx_1 \cdots dx_n$$