Determinants & Cofactors

Last time: we defined determinants using row ops:

(1) If $A \xrightarrow{R_i \mathrel{+}= cR_j} B$ then $\det(A) = \det(B)$.

(2) If $A \xrightarrow{R_i \times = c} B$ then $\det(A) = \frac{1}{c}\det(B)$.

(3) If $A \xrightarrow{R_i \longleftrightarrow R_j} B$ then $\det(A) = -\det(B)$

(4) $\det(I_n) = 1$.

This is the fastest algorithm for computing the det of a general matrix with known entries. But what if the matrix has unknown entries? This becomes tedious because you don't know if an entry is a pivot!

Eg: $\det\begin{pmatrix} -\lambda & 1 & 3 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = ?$ Is $-\lambda$ a pivot?

Cofactor expansion is a handy recursive formula for the determinant that is useful in this setting.

Recursive: Compute $\det(n \times n)$ by computing several $\det((n-1)\times(n-1))$.

Def: Let A be an $n\times n$ matrix.

- The (i,j) minor A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row & j^{th} column.
- The (i,j) cofactor C_{ij} is

$$C_{ij} = (-1)^{i+j}\det(A_{ij})$$

- The cofactor matrix is the matrix C whose (i,j) entry is C_{ij}.

Eg: $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $\quad A_{21} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$

$$C_{21} = (-1)^{2+1}\det\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} = -(-3) = 3$$

NB: $(-1)^{i+j}$ follows a checkerboard pattern: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ $\quad +:\ (-1)^{i+j} = 1$ $\quad -:\ (-1)^{i+j} = -1$

Thm (Cofactor Expansion): A is an $n\times n$ matrix, $a_{ij} = (i,j)$ entry of A, $C_{ij} = (i,j)$ cofactor.

(1) Cofactor expansion along the i^{th} row:

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(2) Cofactor expansion along the j^{th} column:

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Eg: $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

- Expand cofactors along the 3rd row:

$$\det(A) = 1 \cdot \det\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} + 1 \cdot -\det\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} + 0 \det\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= 1 \cdot (1-6) - 1 \cdot (-3) = -2$$

- Expand cofactors along the 2nd column:

$$\det(A) = 1 \cdot -\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + 2 \cdot \det\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} + 1 \cdot -\det\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= 1 \cdot -(-1) + 2 \cdot (-3) + 1 \cdot -(-3) = 1 - 6 + 3 = -2$$

Remarks:

(1) This is a recursive formula: $C_{ij} = \det((n-1)\times(n-1))$

(2) You can compute $C_{ij} = (-1)^{i+j} \det(A_{ij})$ however you like: you'll always get the same number

(3) Expanding along any row or column gives you $\det(A)$ — always the same number.

(4) This is handy when your matrix has unknown entries or a row/col with a lot of zeros — otherwise it's ridiculously slow: $O(n!\cdot n)$.

Eg: $\det\begin{pmatrix} -\lambda & 1 & 3 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$

$$\xrightarrow[\text{1st col}]{\text{expand}} (-\lambda)\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} + 1\cdot -\det\begin{pmatrix} 1 & 3 \\ 1 & -\lambda \end{pmatrix} + 1\cdot\det\begin{pmatrix} 1 & 3 \\ 2-\lambda & 1 \end{pmatrix}$$

$$= -\lambda\big((2-\lambda)(-\lambda)-1\big) + 1\cdot -(-\lambda-3) + 1\cdot\big(1-3(2-\lambda)\big)$$

$$= -\lambda(-2\lambda+\lambda^2-1) + (\lambda+3) + 1-3(2-\lambda)$$

$$= -\lambda^3+2\lambda^2+5\lambda-2$$

In fact, for 3x3 matrices it's not so hard to compute the determinant when all entries are unknown:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a(ei-fh) - b(di-fg) + c(dh-ge)$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

How to remember this?

$\boxed{\begin{array}{l}\textbf{Sarrus' Scheme:} \quad \text{To compute } \det(3\times 3 \text{ matrix}): \\ \begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array} \quad \det = aei + bfg + cdh \\ \qquad\qquad\qquad\qquad\qquad - ceg - afh - bdi \end{array}}$

Sum the products of forward diagonals, subtract products of backwards diagonals.

Eg: $\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 0\cdot 2\cdot 0 + 1\cdot 1\cdot 1 + 3\cdot 1\cdot 1$

$\qquad - 1\cdot 2\cdot 3 - 1\cdot 1\cdot 0 - 0\cdot 1\cdot 1$

$$\begin{matrix} 0 & 1 & 3 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{matrix} \qquad = 4 - 6 = -2$$

Warning: This only works for 3x3 matrices!

→ See the big formula at the end for nxn matrices.

Eg: $\det\begin{pmatrix} 2 & 5 & -3 & -1 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}$ column with lots of zeros

$$= -1\cdot -\det\begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} - 5\cdot\det\begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix}$$

$$+0\cdot -\det\begin{pmatrix}\text{don't} \\ \text{care}\end{pmatrix} + 0\cdot\det\begin{pmatrix}\text{don't} \\ \text{care}\end{pmatrix}$$

$$= 1(-24) - 5(11) = -24 - 55 = -79$$

only computed two 3x3 dets

Better: Do a row operation first!

$$\det\begin{pmatrix} 2 & 5 & -3 & -1 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix} \xrightarrow{R_2 - 5R_1} \det\begin{pmatrix} 2 & 5 & -3 & -1 \\ -12 & -28 & 17 & 0 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}$$

$$= -1\cdot -\det\begin{pmatrix} -12 & -28 & 17 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} = -79$$

only computed one 3x3 det

Methods for Computing Determinants

(1) Special formulas (2x2, 3x3)

→ best for small matrices, except 3x3 with lots of 0's

(2) Cofactor expansion

→ best if you have unknown entries, or a row/column with lots of zeros.

(3) Row (& column) operations

→ best if you have a big matrix with no unknown entries & no row or column with lots of zeros.

(4) Any combination of the above

→ eg. do a row op. to create a column with lots of zeros, then expand cofactors, ...

Thm: Let C be the cofactor matrix of A. Then

$$AC^T = \det(A) I_n = C^T A$$

In particular, if $\det(A) \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{\det(A)} C^T}$$

see supplement

⇝ Ridiculously inefficient computationally.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

⇝ generalizes the formula for 2x2 inverse

Cross Products

This is an operation you can do to vectors in $\mathbb{R}^3$.

Recall: the unit vectors in $\mathbb{R}^3$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Def: Let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $w = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \in \mathbb{R}^3$.

The cross product is

$$v \times w = \begin{pmatrix} bf - ec \\ cd - af \\ ae - bd \end{pmatrix} \in \mathbb{R}^3$$

So the cross product is $(\text{vector}) \times (\text{vector}) \rightsquigarrow (\text{vector})$

Here's how you remember it:

expand cofactors

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \text{"}\det \begin{pmatrix} e_1 & e_2 & e_3 \\ a & b & c \\ d & e & f \end{pmatrix}\text{"}$$

$$= e_1 \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} - e_2 \det \begin{pmatrix} a & c \\ d & f \end{pmatrix} + e_3 \det \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

$$= (bf - ec)e_1 - (af - cd)e_2 + (ae - bd)e_3$$

$$= \begin{pmatrix} bf - ec \\ cd - af \\ ae - bd \end{pmatrix}$$

Eg: $\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\0\end{pmatrix} = \text{"}\det\begin{pmatrix}e_1 & e_2 & e_3\\1 & 2 & 1\\1 & 1 & 0\end{pmatrix}\text{"}$

$$= e_1\det\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix} - e_2\det\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix} + e_3\det\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}$$

$$= -e_1 + e_2 - e_3 = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

Def: Let $u,v,w \in \mathbb{R}^3$. The **triple product** is

$$u\cdot(v\times w) = \det\begin{pmatrix}— u^T —\\— v^T —\\— w^T —\end{pmatrix}$$

Check: if $v=(a,b,c)$ $w=(d,e,f)$ $u=(g,h,i)$ then

$$u\cdot(v\times w)$$

$$= \begin{pmatrix}g\\h\\i\end{pmatrix}\cdot\left(e_1\det\begin{pmatrix}b & c\\e & f\end{pmatrix} - e_2\det\begin{pmatrix}a & c\\d & f\end{pmatrix} + e_3\det\begin{pmatrix}a & b\\d & e\end{pmatrix}\right)$$

$$= g\det\begin{pmatrix}b & c\\e & f\end{pmatrix} - h\det\begin{pmatrix}a & c\\d & f\end{pmatrix} + i\det\begin{pmatrix}a & b\\d & e\end{pmatrix}$$

$$= \det\begin{pmatrix}g & h & i\\a & b & c\\d & e & f\end{pmatrix} \quad ✓$$

Eg: $u=\begin{pmatrix}0\\1\\3\end{pmatrix}$ $v=\begin{pmatrix}1\\2\\1\end{pmatrix}$ $w=\begin{pmatrix}1\\1\\0\end{pmatrix}$

$$v\times w = \begin{pmatrix}-1\\1\\-1\end{pmatrix} \qquad u\cdot(v\times w) = \begin{pmatrix}0\\1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix} = 1-3 = -2$$

$$\det\begin{pmatrix}0 & 1 & 3\\1 & 2 & 1\\1 & 1 & 0\end{pmatrix} = -2$$

Properties:

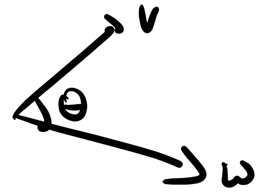
(1) $v \times w \perp v$ and $v \times w \perp w$

$\rightarrow$ because $v \cdot (v \times w) = \det \begin{pmatrix} - v^T - \\ - v^T - \\ - w^T - \end{pmatrix} = 0$

(2) $w \times v = - v \times w$

$\rightarrow$ because $\det \begin{pmatrix} e_1 & e_2 & e_3 \\ - & w^T & - \\ - & v^T & - \end{pmatrix} \overset{\text{row swap}}{=} -\det \begin{pmatrix} e_1 & e_2 & e_3 \\ - & v^T & - \\ - & w^T & - \end{pmatrix}$

(3) $\|v \times w\| = \|v\| \cdot \|w\| \cdot \sin(\theta)$

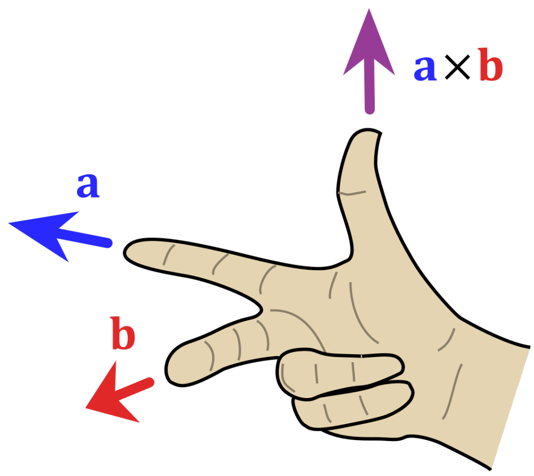


$\rightarrow$ compare $v \cdot w = \|v\| \cdot \|w\| \cdot \cos(\theta)$

(4) $v \times w = 0 \iff v, w$ are collinear

(then $\theta = 0$ or $180^\circ \iff \sin(\theta) = 0$)

(5) $v \times w$ points in the direction determined by the right hand rule.



NB: (1), (3), & (5) characterize $v \times w$.

The Big Formula

This is an explicit formula for $\det(A)$.
It's useful for some things but not practical — it has $n!$ terms!

Def: A permutation of $\{1, \ldots, n\}$ is a re-ordering
$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$$
$\sigma(i)$ = new number in i^{th} position

Eg:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{array} \;\sigma$$

$\sigma(1) = 3$ $\quad \sigma(3) = 2$
$\sigma(2) = 1$ $\quad \sigma(4) = 4$

Q: how many permutations of $\{1, \ldots, n\}$ are there?

- n choices for 1^{st} spot
- $(n-1)$ choices for 2^{nd} spot

  ⋮

- 1 choice for last spot

So $n \cdot (n-1) \cdot \cdots (1) = n!$

Eg: $n=3$: $123 \longrightarrow$

$$\underbrace{123 \quad 132 \quad 213 \quad 231 \quad 312 \quad 321}_{6 = 3 \cdot 2 \cdot 1 = 3!}$$

Def: A transposition is a permutation that just swaps two numbers.

Eg: 123 ⟶ 132 213 321

Fact: Any permutation can be obtained by doing some number of transpositions.

Def: The sign of a permutation σ is $\text{sign}(\sigma) =$

- $+1$ if it can be obtained by doing an even number of transpositions.
- -1 if it can be obtained by doing an odd number of transpositions.

Eg: 1234 ⟶ 3124:

1234 ⟶ 3214 ⟶ 3124

2 transpositions ⟹ sign is $+1$.

Thm (Big Formula): Let A be an $n \times n$ matrix with (i,j) entry a_{ij}.

$$\det(A) = \sum_{\text{permutations } \sigma} \text{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Eg: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Six permutations: $123 \to$

1 2 3	sign = 1	(0 transpositions)
1 3 2	sign = −1	(transposition)
2 1 3	sign = −1	(transposition)
2 3 1	sign = 1	(2 transpositions)
3 2 1	sign = −1	(transposition)
3 1 2	sign = 1	(2 transpositions)

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}$$

✓