Solving Systems of Equations using Elimination

Here's a system of 3 equations in 3 variables:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases}$$

How to solve it?

- **Substitution:** solve 1st equation for x_1, substitute into 2nd & 3rd, continue.
- **Elimination:** "combine" the equations to eliminate variables.

Elimination turns out to scale much better (to more equations & variables), so we'll focus on that.

"replace the 2nd equation with the 2nd minus 2x the 1st"

Eg:

$$\begin{matrix} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{matrix} \quad \xrightarrow{R_2 -= 2R_1} \quad \begin{matrix} x_1 + 2x_2 + 3x_3 = 6 \\ -7x_2 - 4x_3 = 2 \\ 3x_1 + x_2 - x_3 = -2 \end{matrix}$$

$$\xrightarrow{R_3 -= 3R_1} \quad \begin{matrix} x_1 + 2x_2 + 3x_3 = 6 \\ -7x_2 - 4x_3 = 2 \\ -5x_2 - 10x_3 = -20 \end{matrix}$$

Now we have eliminated x_1 from the 2nd & 3rd eq.s

These now form 2 equations in 2 variables: simpler!

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -5x_2 - 10x_3 &= -20 \end{aligned} \quad \xrightarrow{R_3 - \frac{5}{7}R_2} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -\tfrac{50}{7}x_3 &= -\tfrac{150}{7} \end{aligned}$$

isolated

We eliminated x_2 from the last equation: now it's one equation in one variable. Easy!

We can now solve via back-substitution:

$$-\frac{50}{7}x_3 = -\frac{150}{7} \Rightarrow \boxed{x_3 = 3}.$$

Substitute into 2nd equation:

$$-7x_2 - 4x_3 = 2 \rightsquigarrow -7x_2 - 4 \cdot 3 = 2$$

isolated

Now solve for x_2:

$$-7x_2 - 12 = 2 \Rightarrow -7x_2 = 14 \Rightarrow \boxed{x_2 = -2}$$

Substitute both into 1st equation:

$$x_1 + 2x_2 + 3x_3 = 6 \rightsquigarrow x_1 + 2 \cdot (-2) + 3 \cdot 3 = 6$$

isolated

Now solve for x_1:

$$x_1 - 4 + 9 = 6 \Rightarrow \boxed{x_1 = 1}$$

Check:

$$\begin{aligned} 1 + 2(-2) + 3(3) &= 6 \\ 2 \cdot 1 - 3(-2) + 2(3) &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned} \quad \checkmark$$

NB: In this case there was one solution — since we could isolate each variable, all values were determined.

Does this always work?

Eg: $4x_2 + 3x_3 = 2$ swap
$x_1 + x_2 - x_3 = 3$
$2x_1 - 3x_2 - 6x_3 = -3$

x_1 is already eliminated from R_1. Fix: swap the 1st 2 eqns.

$$\xrightarrow{R_1 \leftrightarrow R_2} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ 2x_1 - 3x_2 - 6x_3 &= -3 \end{aligned}$$

Now eliminate as before:

$$\xrightarrow{R_3 -= 2R_1} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ -5x_2 - 4x_3 &= -9 \end{aligned}$$

$$\xrightarrow{R_3 += \frac{5}{4}R_2} \quad \begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ -\tfrac{1}{4}x_3 &= -\tfrac{13}{2} \end{aligned}$$

Solve using back-substitution:

isolated $\quad -\frac{1}{4}x_3 = -\frac{13}{2} \implies x_3 = 26$

Substitute into 2nd equation:

isolated $\quad 4x_2 + 3(26) = 2 \implies x_2 = -19$

Substitute both into 1st equation:

isolated $\quad x_1 - 19 - 26 = 3 \implies x_1 = 48$

NB again there is one solution: each variable was isolated in one equation.

Check:
$$4x_2 + 3x_3 = 2 \qquad 4(-19)+3(26)=2$$
$$x_1 + x_2 - x_3 = 3 \quad \rightsquigarrow \quad 48 - 19 - 26 = 3$$
$$2x_1 - 3x_2 - 6x_3 = -3 \qquad 2(48)-3(-19)-6(26) = -3$$
✓

Eg:
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= -1 \end{aligned} \quad \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ -6x_2 - 12x_3 &= -8 \end{aligned}$$

can't isolate x_3!

$$\xrightarrow{R_3 -= 2R_2} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 &= 0 \end{aligned}$$

Are we done? Yes: choose any value for x_3, then back-substitute to find x_1, x_2:

$$-3x_2 = -4 + 6x_3 \implies x_2 = \frac{4}{3} - 2x_3$$
$$x_1 = 1 - 2x_2 - 3x_3 = 1 - \frac{8}{3} + 4x_3 - 3x_3$$
$$x_1 = -5/3 + x_3$$

Eg: $x_3 = 1 \rightsquigarrow x_1 = -2/3,\ x_2 = -2/3$

Check:
$$-2/3 - 4/3 + 3 = 1$$
$$-8/3 - 10/3 + 6 = 0$$
$$-14/3 - 16/3 + 9 = -1$$
✓

In this case there are infinitely many solutions. We'll deal with this in Week 3.

Eg: $x_1 + 2x_2 + 3x_3 = 1$ $\quad$ $\xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1}$ $\quad$ $x_1 + 2x_2 + 3x_3 = 1$

$4x_1 + 5x_2 + 6x_3 = 0$ $\qquad\qquad\qquad\qquad$ $-3x_2 - 6x_3 = -4$

$7x_1 + 8x_2 + 9x_3 = 0$ $\qquad\qquad\qquad\qquad$ $-6x_2 - 12x_3 = -7$

tweak previous example

$\xrightarrow{R_3 -= 2R_2}$ $\quad$ $x_1 + 2x_2 + 3x_3 = 1$

$-3x_2 - 6x_3 = -4$

$0 = 1$

If our original equations were true, then $0 = 1$.

Thus our system has no solutions.

(last 2 eqns are parallel planes)

Row Operations are the allowed manipulations we can perform on our equations.

(1) $x_1 + 2x_2 + 3x_3 = 6$ $\quad$ $\xrightarrow{R_2 -= 2R_1}$ $\quad$ $x_1 + 2x_2 + 3x_3 = 6$

$2x_1 - 3x_2 + 2x_3 = 14$ $\qquad\qquad$ $-7x_2 - 4x_3 = 2$

$3x_1 + x_2 - x_3 = -2$ $\qquad\qquad$ $3x_1 + x_2 - x_3 = -2$

row replacement

replace R_2 by $R_2 - 2R_1$

(2) $x_1 + 2x_2 + 3x_3 = 6$ $\quad$ $\xrightarrow{R_1 \leftrightarrow R_2}$ $\quad$ $2x_1 - 3x_2 + 2x_3 = 14$

$2x_1 - 3x_2 + 2x_3 = 14$ $\qquad\qquad$ $x_1 + 2x_2 + 3x_3 = 6$

$3x_1 + x_2 - x_3 = -2$ $\qquad\qquad$ $3x_1 + x_2 - x_3 = -2$

row swap

(change order)

(3)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \quad \xrightarrow{R_1 \times = 2} \quad \begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 12 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

scalar multiplication
(by nonzero scalar)

Obviously if (x_1, x_2, x_3) is a solution before doing a row operation, then it is true after. Eg. row replacement:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \to 6 &= 6 \\ 2x_1 - 3x_2 + 2x_3 \to 14 &= 14 \end{aligned} \quad \xrightarrow{R_2 -= 2R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 \to 6 &= 6 \\ -7x_2 - 4x_3 \to 2 &= 2. \end{aligned}$$

Fact: All these operations are reversible: if you have a solution (x_1, x_2, x_3) after doing a row operation, then it's also a solution before.

(system of equations) $\xrightarrow{\text{row operations}}$ ((simpler) system of equations)

SAME SOLUTIONS

This was the whole point: we wanted to solve our (original) system of equations!

Questions: How do you undo (reverse):

- $R_1 += R_2$? $R_1 -= R_2$
- $R_1 \times = 2$? $R_1 \div = 2$
- $R_1 \leftrightarrow R_2$? $R_1 \leftrightarrow R_2$

The variables $x_1, x_2, \ldots$ are just placeholders; only their coefficients matter. Let's extract them into a matrix.

Three Ways to Write System of Linear Equations

(1) As a system of equations:

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$

(2) As a matrix equation $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 14 \end{bmatrix}}_{b}$$

If you expand out the product you get

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

which is what we had before.

The coefficient matrix A comes from the coefficients of the variables:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix} \longleftrightarrow \begin{matrix} 1x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{matrix}$$

The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ contains the unknowns or variables.

NB: A is an $m \times n$ matrix where

m = # equations $\qquad b \in \mathbb{R}^m$ ← size m

n = # variables $\qquad x \in \mathbb{R}^n$ ← size n

(3) As an augmented matrix.

This is a notational convenience: just squash A & b together and separate with a line.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \longleftrightarrow \begin{array}{l} 1x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \end{array}$$

$$\| $$

$$[A \mid b]$$

Augmented matrices are good for row operations, which only affect the coefficients (not the variables):

$$\begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \end{array} \underset{\leadsto}{R_2 -= 2R_1} \begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ \quad -7x_2 - 4x_3 = 2 \end{array}$$

$$\equiv$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \underset{\leadsto}{R_2 -= 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \end{array}\right]$$

Eg: Let's solve the system from before using augmented matrices:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \xrightarrow{R_2 -= 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 -= 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$$

$$\xrightarrow{R_3 -= \frac{5}{7}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

$$\rightsquigarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ -7x_2 - 4x_3 = 2 \\ -\frac{50}{7}x_3 = -\frac{150}{7} \end{cases}$$

Now use back-substitution like before.

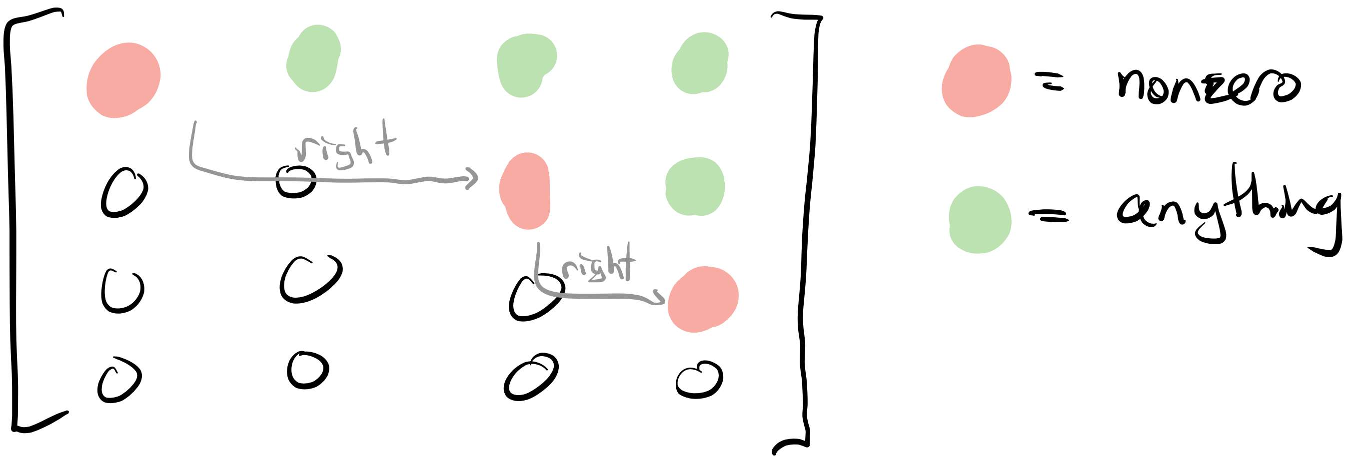
What does it mean to be "done"?

(in terms of augmented matrices)

Def: A matrix is in **row echelon form (REF)** if

(1) The first nonzero entry of each row is to the right of the row above it

(2) All zero rows are at the bottom



REF:
$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

Not REF:
$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Important: When checking if an augmented matrix is in REF, ignore the augmentation line.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right] \text{ REF? } \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \checkmark$$

← delete

Think: REF means there's nothing left to eliminate! Each variable is eliminated in later equations, or can't be isolated.

Upshot: The elimination procedure terminates when your (augmented) matrix is in REF.

Solving a linear system $\equiv$ Putting an augmented matrix into REF using row operations

Def: The pivot positions (pivots) of a matrix are the positions of the 1^{st} nonzero entries of each row after you put it into REF.

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

= pivots

Remarkably, this is well-defined!

Def: The rank of a matrix is the number of pivots it has (in REF).

Eg:
$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \xrightarrow[\text{(p.9)}]{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

rank = 3

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{bmatrix} \xrightarrow[\text{(p.4)}]{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2

Number of Solutions (in terms of pivots)

The most basic question you can ask about a system of equations is: how many solutions does it have? This is entirely determined by the pivot positions/pivot columns (columns with a pivot).

(1) The system

pivot columns

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

(p.2)

had one solution. It has a pivot in every column except the augmented column. This means every variable will be isolated when doing back-substitution.

(0) The system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(p.5)

had no solutions. It has a pivot in the augmented column, which leads to the equation $0=1$.

(∞) The system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(p. 4)

had infinitely many solutions. It has no pivot in the augmented column and no pivot in the column for the variable x_3. You can't isolate x_3, so you can choose any value.

NB: You have to put the system in REF to find its pivots, so you have to do work to know how many solutions there are.

Def: A system is consistent if it has at least 1 solution (so 1 or ∞). It is inconsistent otherwise.