# Elementary Matrices

These give a way to do row operations by matrix multiplication!

**Def:** An *elementary matrix* is a matrix obtained from $I_n$ by doing *one* row operation.

**Eg:**

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mathrel{+}= 2R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \times= 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Fact:** If $E$ is an $m \times m$ elementary matrix and $A$ is an $m \times n$ matrix, then

$$E \cdot A = \left(\begin{array}{l}\text{what you get by performing that} \\ \text{row operation on } A.\end{array}\right)$$

$$\left(\text{row operations}\right) \equiv \left(\begin{array}{l}\text{left-multiplication by} \\ \text{elementary matrices}\end{array}\right)$$

Eg: $\begin{bmatrix} -2 & -3 & 0 & -7 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mathrel{+}= 2R_2} \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mathrel{+}= 2R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ same!

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 0 & -7 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Left-multiplication by $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ does $R_1 \mathrel{+}= 2R_2$.

**Fact:** An elementary matrix is invertible. Its inverse corresponds to the elementary matrix that un-does the row operation.

Why? $E_1 = (\text{matrix for } R_1 \mathrel{+}= 2R_2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = (\text{matrix for } R_1 \mathrel{-}= 2R_2) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_1E_2 = E_1E_2I_n = E_1(\text{do } R_1 \mathrel{-}= 2R_2 \text{ to } I_n)$

$= \left(\begin{array}{l} \text{first do } R_1 \mathrel{-}= 2R_2 \text{ to } I_n, \\ \text{then do } R_1 \mathrel{+}= 2R_2 \text{ to the result} \end{array}\right) = I_n.$

Check: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

# What if you do multiple row operations?

Consider these row operations & their elementary matrices:

$E_1$: $R_1 \mathrel{+}= 2R_2$ $\quad$ $E_2$: $R_2 \times= 2$ $\quad$ $E_3$: $R_2 \leftrightarrow R_3$

Apply in order to $A$

$$A \xrightarrow{R_1 \mathrel{+}= 2R_2} E_1A \xrightarrow[\text{do } R_2 \times= 2 \text{ to } E_1A]{R_2 \times= 2} E_2(E_1A) \xrightarrow{R_2 \leftrightarrow R_3} E_3(E_2(E_1A))$$

$$= (E_3E_2E_1)A$$

The elementary matrices ended up in the opposite order! Why?

$E_3E_2E_1A = E_3E_2(E_1A)$: first multiply by $E_1$ then by $E_2$ then $E_3$

# Application to Invertibility:

Suppose $\text{RREF}(A) = I_n$.

So there are some number of row ops to transform $A \rightsquigarrow I_n$.

Let $E_1, \dots, E_r$ be their elementary matrices.

$$I_n = (E_r E_{r-1} \cdots E_1) A$$

$$\Rightarrow \boxed{A^{-1} = E_r E_{r-1} \cdots E_1}$$

In particular, $A$ is invertible: justifies (part of) the thm last time.

This also justifies the algorithm for computing $A^{-1}$:

$$[A \mid I_n] \xrightarrow{\text{row ops}} [I_n \mid B]$$

Then $[I_n \mid B] = (E_r \cdots E_1)[A \mid I_n]$

$$= [(E_r \cdots E_1)A \mid (E_r \cdots E_1)]$$

$\Rightarrow B = E_r \cdots E_1 = A^{-1}$

column-1st matrix multiplication

$$C[A|B] = [CA|CB]$$

# Triangular Matrices

Will lead to LU decomposition ~ computationally efficient way to solve $Ax=b$ for many values of $b$.

**Def:** A matrix is **upper/lower triangular** if all entries below/above the diagonal are zero.

upper triangular

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \end{bmatrix}$$

$\bullet$ = anything

A matrix is **unitriangular** if it is triangular and all diagonal entries = 1.

upper unitriangular

$$\begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \end{bmatrix}$$

lower unitriangular

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \bullet & 1 & 0 & 0 \\ \bullet & \bullet & 1 & 0 \end{bmatrix}$$

**NB:** A matrix is **diagonal** $\Leftrightarrow$ if & both upper- and lower- triangular.

**Eg:** A matrix in REF is upper-$\Delta$ular

$$\begin{pmatrix} 1 & 7 & 2 & 4 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Eg:** If $E$ is the elementary matrix for $R_i \mathrel{+}= c \cdot R_j$ for $i \geq j$ (add a higher row to a lower row) then $E$ is lower-unitriangular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \mathrel{+}= 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E$$

**Fact:** If $A$ & $B$ are $n \times n$ upper (uni) triangular matrices then so are $AB$ and $A^{-1}$ (if $A$ is invertible).

Likewise for lower (uni) triangular.

**Eg:**
$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 29 & 11 & 1 \end{pmatrix}$$

**NB:** Unitriangular & square $\Rightarrow$ invertible

# LU Decompositions

**Fact:** If Gaussian elimination on $A$ requires no row swaps, then

$$A = LU$$

for $L$ lower-unitriangular and $U$ in REF.

**Eg:** $\begin{pmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 6 & 1 & 0 \end{pmatrix} = A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

## Why do we care?

**Recall:** Solving $Ax=b$ (for $A$ $n \times n$) requires $\approx \frac{2}{3}n^3$ flops

## Algorithm (solving $Ax=b$ using $A=LU$)

**Input:** An LU factorization $A=LU$ & vector $b$

**Output:** A solution of $Ax=b$

**Procedure:**

(1) Solve $Ly=b$ using forward-substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ \bullet & 1 & 0 \\ \bullet & \bullet & 1 \end{bmatrix} y = b \equiv \begin{aligned} y_1 &= b_1 \\ y_2 + \bullet y_1 &= b_2 \\ y_3 + \bullet y_2 + \bullet y_1 &= b_3 \end{aligned}$$

($n^2 - n$ flops)

(2) Solve $Ux = y$ using backward-substitution ($n^2$ flops)

Then $Ax = LUx = L(Ux) = Ly = b$ ✓

NB: total complexity is $\approx 2n^2$ flops
→ way faster than $\frac{2}{3}n^3$!

Eg: Solve $Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ using

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 6 & 1 & 0 \end{bmatrix} = A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$$

(1) $Ly = b$: $\begin{aligned} y_1 &= 1 \\ 2y_1 + y_2 &= 0 \\ 3y_1 - y_2 + y_3 &= 1 \end{aligned}$ forward subst $\Rightarrow y = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$

(1) $Ux = y$: $\begin{aligned} 2x_1 + x_2 &= 1 \\ 2x_2 + 4x_3 &= -2 \\ 4x_3 &= -4 \end{aligned}$ backward subst $\Rightarrow x = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Check: $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 4 & 4 \\ 6 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ✓

# Where does $A = LU$ come from? How to compute it?

If you can run Gaussian elimination with no row swaps then $A \xrightarrow[\text{ops}]{\text{row}} U = \text{REF}$ using row ops of the form

$$R_i \mathrel{+}= cR_j \quad (i \neq j) \longleftrightarrow E = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ c & & 1 \end{bmatrix} \quad R_3 \mathrel{+}= cR_1$$

In this case $U = (E_r \cdots E_1)A$, $U = \text{REF}$

$E_i$ = elementary matrix for $R_i \mathrel{+}= cR_j$, $i > j$ (clear down)

$E_i$ is lower-unitriangular

$\Rightarrow E_r \cdots E_1$ is lower-unitriangular

$\Rightarrow L = (E_r \cdots E_1)^{-1}$ is lower-unitriangular

and $LU = \underbrace{(E_r \cdots E_1)^{-1}}_{L} \underbrace{(E_r \cdots E_1)A}_{U} = I_n A = A$ ✓

NB: $L = (E_r \cdots E_1)^{-1}$ "records the row operations"
→ keeps track of elimination

NB: $A = LU$ is a matrix factorization: it is a way to write a matrix as a product of simpler matrices. We will learn many more of these.

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 -= 4R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$ $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 -= 7R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 -= 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$U$

$L = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$

To compute $E_1^{-1} E_2^{-1} E_3^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} I_3$,

start with $I_3$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 += 2R_2}$ $E_3^{-1}$: undo $E_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$\xrightarrow{R_3 += 7R_1}$ $E_2^{-1}$: undo $E_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$

$\xrightarrow{R_2 += 4R_1}$ $E_1^{-1}$: undo $E_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} = L$

Check: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

Here's a better way to do the bookkeeping.

## Algorithm (LU Decomposition; 2 column method):

**Input:** A matrix where Gaussian elimination requires no row swaps

**Output:** A factorization $A=LU$ for $L$ lower-unitriangular and $U$ in REF (the output of Gaussian elimination).

**Procedure:** Do Gaussian elimination, keeping track of the row operations as follows: start with a blank $m \times m$ matrix $L$.

- for each row replacement $R_i \mathrel{+}= c R_j$, put $-c$ in the $(i,j)$ entry of $L$.

Add 1's & 0's to the remaining blank entries of $L$ (1's on the diagonal, 0's elsewhere). Then

$$A = LU.$$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

2 columns

| L | U | |
|---|---|---|
| $\begin{bmatrix} \text{blank} \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ | start with A in U column |
| $R_2 -= 4R_1$ $\begin{bmatrix} 4 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$ | |
| $R_3 -= 7R_1$ $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ | |
| $R_3 -= 2R_2$ $\begin{bmatrix} 4 \\ 7 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$ | |

add these

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \qquad U$$

NB: Finding $A = LU$ is just Gaussian elimination + extra bookkeeping $\rightsquigarrow$ same complexity. $\approx \frac{2}{3}n^3$ flops

Then after computing $A = LU$, solving $Ax = b$ requires $\approx 2n^2$ flops.

Still have to do elimination once, but then solving $Ax = b$ for new values of $b$ is much faster.

Eg: If $A$ is $1000 \times 1000$ and we want to solve $Ax=b$ for 1000 values of $b$:

- $\frac{2}{3}$ gigaflops to compute $A=LU$
- 2 megaflops $\times 1000 = 2$ gigaflops to solve $Ax=b$ 1000 times.

That's 250× faster than $\frac{2}{3}$ teraflops from doing elimination 1000 times!

[demo]

## What about inverses?

Wouldn't it be better to compute $A^{-1}$ and solve $A^{-1}x=b$ 1000 times?

No, for 2 reasons:

(1) Computing $A^{-1}$ takes $\approx \frac{4}{3}n^3$ flops: twice as long as the elimination step!

(2) Computing $A^{-1}$ is not numerically stable (less accurate due to rounding errors).

```
from sympy import *
from time import time
 # This is the 10x10 matrix with 2's on the diagonal and 1's elsewhere
 #    eye(n) = nxn identity matrix; ones(n) = nxn matrix of 1's
 # (multiply by 1.0 to force it to use floating point arithmetic)
A = (eye(10) + ones(10)) * 1.0
 # This is the vector [1,1,1,1,1,1,1,1,1,1]
b = ones(10, 1) * 1.0
start = time()
 # Compute LU decomposition
L, U, _ = A.LUdecomposition()
 # Solve using forward- and reverse-substitution 1000 times
for _ in range(1000): U.upper_triangular_solve(L.lower_triangular_solve(b))
end = time()
print(end - start)
 # "7.144780397415161" (seconds)

start = time()
 # Solve using elimination 1000 times
for _ in range(1000): A.solve(b)
end = time()
print(end - start)
 # "48.048372983932495" (seconds)
 # Roughly 10x slower!
```

# Maximal Partial Pivoting

Eg: $\begin{cases} x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$ has one solution: $x_1 = x_2 = 1$

Let's tweak it a little bit:

$\begin{cases} 10^{-17}x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$ $\rightsquigarrow$ presumably has one solution $(x_1, x_2) \approx (1, 1)$.

$$\left[\begin{array}{cc|c} 10^{-17} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \mathrel{-}= 10^{17} R_1} \left[\begin{array}{cc|c} 10^{-17} & 1 & 1 \\ 0 & 1-10^{17} & 2-10^{17} \end{array}\right]$$

$$10^{-17}x_1 + x_2 = 1$$

$\rightsquigarrow$

$$(1-10^{17})x_2 = 2 - 10^{17}$$

$$\Rightarrow x_2 = \frac{2-10^{17}}{1-10^{17}} = \frac{1+(1-10^{17})}{1-10^{17}} = 1 + \frac{1}{1-10^{17}} \approx 1$$

$$x_1 = 10^{17}(1-x_2) = \frac{10^{17}}{10^{17}-1} \approx 1$$

Let's try this on a computer. [demo]

What went wrong?

→ Most programming languages use 64-digit floating point numbers. That means it has $\approx 16$ digits of precision.

The computer thinks $1 - 10^{17} = -10^{17} = 2 - 10^{17}$

```
from sympy import *
 # 1e-17 is 10^(-17)
A = Matrix([[1e-17, 1.0, 1.0], [1.0, 1.0, 2.0]])
 # This does R2 -= 10^(17) R1
 # (force sympy to use the smaller pivot)
A.row_op(1, lambda v, j: v - 1e17*A[0,j])
pprint(A)
 #   [1e-17    1     1]
 #   [0    -1e17 -1e17]
 # Now do Jordan substitution
pprint(A.rref(pivots=False))
 #   [1 0 0]
 #   [0 1 1]
 # This answer is numerically very wrong!
```

$$\left[\begin{array}{cc|c} 10^{-17} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \mathrel{-}= 10^{17} R_1} \left[\begin{array}{cc|c} 10^{-17} & 1 & 1 \\ 0 & -10^{17} & -10^{17} \end{array}\right]$$

$$\rightsquigarrow \quad 10^{-17} x_1 + x_2 = 1$$
$$x_2 = 1$$

$$\rightsquigarrow \quad x_1 = 10^{17}(0) = 0$$

$(0, 1) \not\approx (1, 1)$ : not numerically stable.

**What went wrong?**

Dividing by $10^{-17}$ produced a huge number $10^{17}$
$\rightsquigarrow$ all errors just exploded!

> On a computer, you never want to divide by tiny numbers!

**Solution:** Use the larger pivot (in absolute value)

$$\left[\begin{array}{cc|c} 10^{-17} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-17} & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \mathrel{-}= 10^{-17} R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$x_1 + x_2 = 2$
$x_2 = 1$ $\rightsquigarrow x_1 = 1$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

Computer:
$1 - 10^{-17} = 1$
$1 - > 10^{17} = 1$

## Gaussian Elimination using Maximal Partial Pivoting:

In step (a), row swap so that the largest number in the column (in absolute value) is the pivot.

This is much more numerically stable.
→ avoids dividing by tiny numbers.

## PA=LU Decompositions

Recall: LU only works when you don't need to row swap when eliminating.

But elimination works much better with row swaps! Need to tweak LU.

Idea: Do all the row swaps you need first, then do elimination without row swaps.

(How do you know in advance which row swaps to do? You don't - need to do more bookkeeping)

Def: A permutation matrix is a product of elementary matrices for row swaps.

**PA=LU Decomposition:** Any matrix $A$ has a factorization

$$PA = LU$$

Do a bunch of row swaps on $A$...

... then do an LU decomp

where:

- $P$: permutation matrix
- $L$: lower-unitriangular matrix
- $U$: REF matrix

**Eg:**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix}$$

maximal partial pivoting

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} -10 & -20 & -30 \\ 1 & 1 & 1 \\ 5 & 15 & 10 \end{bmatrix} \quad P_1$$

$R_2 \mathrel{+}= \frac{1}{10} R_1$

$R_3 \mathrel{+}= \frac{1}{2} R_1$

$$\begin{bmatrix} -10 & -20 & -30 \\ 0 & -1 & -2 \\ 0 & 5 & -5 \end{bmatrix} \quad E_1, E_2$$

maximal partial pivoting

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & -1 & -2 \end{bmatrix} \quad P_2$$

$R_3 \mathrel{+}= \frac{1}{5} R_2$

$$\begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix} \quad E_3$$

$$= U$$

with $P_2$ were here...

$$U = E_3 P_2 E_2 E_1 P_1 A$$

not lower-uniΔular!

Trick: $P_2 E_2 E_1 = (P_2 E_2 E_1 P_2) P_2$ ($P_2 P_2 = I_3$)

($P_2 E_2 E_1 P_2$ is lower-unitriangular!)

$$P_2 E_2 E_1 P_2 = P_2 E_2 E_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_3 \mathrel{+}= \frac{1}{2} R_1]{R_2 \mathrel{+}= \frac{1}{10} R_1} P_2 \begin{bmatrix} 1 & 0 & 0 \\ 1/10 & 0 & 1 \\ 1/2 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/10 & 0 & 1 \end{bmatrix}$$

lower-unitriangular!

Then $U = E_3 (P_2 E_2 E_1 P_2) P_2 P_1 A$

$\Rightarrow PA = LU$

for $P = P_2 P_1$ $L = (E_3 P_2 E_2 E_1 P_2)^{-1}$

Here is an efficient way of doing the bookkeeping.

# Algorithm ($PA=LU$ Decomposition; 3-column method):

**Input:** Any matrix $A$

**Output:** A factorization $PA = LU$ where:

- $P$: permutation matrix
- $L$: lower-unitriangular matrix
- $U$: REF matrix

**Procedure:** Perform Gaussian elimination using any pivoting strategy (eg. maximal partial pivoting). Keep track of row operations as follows: start with a blank matrix $L$ and an identity matrix $P$.

- for each row replacement $R_i \mathrel{+}= c R_j$, put $-c$ in the $(i,j)$ entry of $L$ (as before)
- for each row swap $R_i \leftrightarrow R_j$, swap the corresponding rows of $L$ and $P$.

Add 1's & 0's to the remaining blank entries of $L$ (1's on the diagonal, 0's elsewhere)

Then

$$PA = LU.$$

Eg: $A = \begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix}$

3 columns

P | L | U

start with $I_m$ here → $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \text{blank} \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix}$ ← start with A in the U col

$R_1 \leftrightarrow R_2$ → $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \text{blank} \end{bmatrix}$ $\begin{bmatrix} -10 & -20 & -30 \\ 1 & 1 & 1 \\ 5 & 15 & 10 \end{bmatrix}$

$R_2 \mathrel{+}= \frac{1}{10} R_1$, $R_3 \mathrel{+}= \frac{1}{2} R_1$ → $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} -1/10 \\ -1/2 \end{bmatrix}$ $\begin{bmatrix} -10 & -20 & -30 \\ 0 & -1 & -2 \\ 0 & 5 & -5 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ → $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} -1/2 \\ -1/10 \end{bmatrix}$ $\begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & -1 & -2 \end{bmatrix}$

$R_3 \mathrel{+}= \frac{1}{5} R_2$ → $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} -1/2 \\ -1/10 & -1/5 \end{bmatrix}$ $\begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix}$

P U

(add these?)

$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/10 & -1/5 & 1 \end{bmatrix}$

Check:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/10 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix}$$

Can we still use this to solve $Ax=b$?

## Algorithm (solving $Ax=b$ using $PA=LU$)

Solving $Ax=b$ is the same as $PAx=Pb$, so:

(0) Compute $Pb$ (re-order the entries of $b$) (0 flops)

(1) Solve $Ly=Pb$ using forward-substitution ($n^2-n$ flops)

(2) Solve $Ux=y$ using backward-substitution ($n^2$ flops)

Then $PAx = LUx = Ly = Pb$

$\Rightarrow Ax=b$ (multiply both sides by $P^{-1}$) ✓

(total $\leq n^2$ flops)

**Eg:** Solve $Ax=b$ for

$$A=\begin{bmatrix} 1 & 1 & 1 \\ -10 & -20 & -30 \\ 5 & 15 & 10 \end{bmatrix} \qquad b=\begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$$

(0) $$Pb=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}=\begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix}$$

(1) $$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/10 & -1/5 & 1 \end{bmatrix} y = Pb = \begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix} \rightsquigarrow \begin{matrix} y_1=-10 \\ y_2=5 \\ y_3=0 \end{matrix}$$

(2) $$\begin{bmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{bmatrix} x = \begin{bmatrix} -10 \\ 5 \\ 0 \end{bmatrix} \rightsquigarrow \begin{matrix} x_1=-1 \\ x_2=1 \\ x_3=0 \end{matrix} \qquad x=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$