Orientation: In some sense we've learned to solve a linear system $Ax=b$ using elimination & substitution. But there are some important questions still to answer:

(1) For which vectors b is $Ax=b$ consistent?
(ie for which b does a solution exist?)

(2) If a solution exists, how to describe all solutions of $Ax=b$?

Note that (2) is really only interesting if there are ∞ solutions.

Parametric Form

Now we deal systematically with systems of equations with ∞ solutions. We want to parameterize all solutions.

Eg: $2x+y+12z=1$
$x+2y+9z=-1$ $\rightsquigarrow \left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right]$

$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \rightsquigarrow$ $x+5z=1$
$y+2z=-1$

Observation: If you substitute any number for z, you get the system

unknowns

$$\left[\begin{array}{l} x = 1-5z \\ y = -1-2z \end{array}\right.$$

numbers

which has a unique solution!

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$$

eg $z=1$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$

check: $2(-4)+(-3)+12(1)=1$
$-4+2(-3)+9(1)=-1$ ✓

This is the parametric form of the solution; z is the free variable or parameter.

Implicit vs Parameterized Form.

- The system of equations $\begin{cases} 2x+y+12z=1 \\ x+2y+9z=-1 \end{cases}$

  are implicit equations of a line: it expresses the line as the set of solutions of these equations without giving you any way to write down specific points on the line.

  (Good for checking if (a,b,c) is on the line)

- The parametric form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$$

  is a parametric equation for the same line: it gives you a way to produce all solutions in terms of the parameter z.

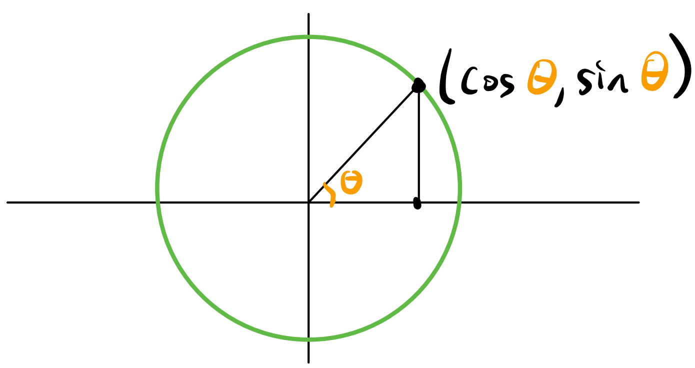
  (Good for producing points on the line)

[demo]

Non-linear example:

An implicit equation for the unit circle is

$$x^2+y^2=1$$



A parametric equation for the unit circle is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \theta = \text{parameter}$$

Here's how to produce parametric equations for general linear systems.

Recall: A pivot column of a matrix is a column with a pivot.

Def: A free variable in a system of equations is a variable whose column (in the coeff matrix) is not a pivot column.

$$\begin{array}{ccc} & & \\ \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

x, y in pivot cols

z is free

These are the variables you can't isolate in back-substitution.

Procedure (Parametric Form):

To find the parametric form of the solutions of $Ax=b$:

(1) Put $[A \mid b]$ into RREF. Stop if inconsistent.

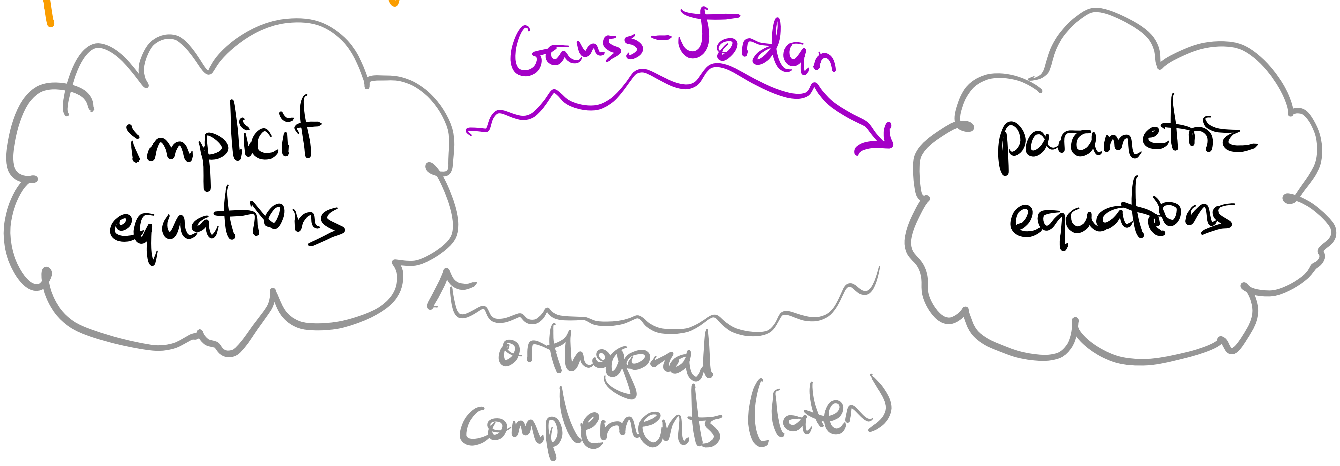
(2) Write out the corresponding equations

(3) Move free variables to the right-hand side

All solutions are obtained by substituting any values for the free variables.

This uses the free variables as the parameters.

So Gauss-Jordan elimination turns implicit equations into parametric equations.
↳ (step (1) above)



Eg: $x+y+z=1 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right]$

This is already in RREF!

Free variables: y, z

$\xrightarrow[\text{form}]{\text{parametric}} \quad x = 1-y-z$

This is a parameterized plane. [demo]

Eg: $\begin{aligned} x+y&=2 \\ x-y&=0 \end{aligned} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$

No free variables! Just have one solution $x=1,\ y=1$.

Observation:

- 2 free variables / 2 parameters:
  solution set is a plane
- 1 free variable / 1 parameter:
  solution set is a line
- 0 free variables / 0 parameters:
  solution set is a point

Provisional Def^n: The dimension of the solution set of a consistent system $Ax=b$ is the number of free variables.

Parametric Vector Form

This is an alternate, more concise way of writing a solution set in parametric form.

Eg: $\begin{aligned} 2x+y+12z &= 1 \\ x+2y+9z &= -1 \end{aligned} \xrightarrow[\text{form}]{\text{parametric}} \begin{aligned} x &= 1-5z \\ y &= -1-2z \\ z &= z \end{aligned}$

write in columns (from before)

parameterize the free variable too

Let's rewrite this as one equation involving vectors:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \boxed{z \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}}$$

linear combination of $(-5,-2,1)$

This is the line thru $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ in the $\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$-direction. [demo again]

write in columns

Eg: $x+y+z=1 \xrightarrow[\text{form}]{\text{parametric}} \begin{aligned} x &= 1-y-z \\ y &= y \\ z &= z \end{aligned}$

parameterize the free variables

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \boxed{y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

linear combination of $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

This is the plane containing $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, & $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

[demo again]

Writing the solution set in this way is called the parametric vector form.

Procedure (Parametric Vector Form)

To find the parametric vector form of the solutions of $Ax=b$:

(1 – 3) Find the parametric form

(4) Add trivial equations for the free variables, in order. Organize the right-hand side into columns.

(5) Gather the columns into vectors. Pull out the free variables as coefficients.

Result:

$$x = \left(\text{a constant vector}\right) + \left(\text{a linear combination with the free variables as weights}\right)$$

NB: The constant vector is the solution you get by setting all free variables $=0$.

Def: This vector is called a particular solution. (It is a solution of $Ax=b$)

Eg:
$$\begin{aligned} x + 2y + 2z + w &= 1 \\ 2x + 4y + z - w &= -1 \end{aligned} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\rightsquigarrow \begin{aligned} x + 2y - w &= -1 \\ z + w &= 1 \end{aligned}$$

free

$$\rightsquigarrow \begin{cases} x = -1 - 2y + w \\ y = y \\ z = 1 - w \\ w = w \end{cases}$$

trivial equations

columns

$$\overset{\text{PVF}}{\rightsquigarrow} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\text{any linear combination of } \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}$$

Vector Equations

This is another way of writing a linear system that works well with what we've been doing.

Def: A **vector equation** is an equation of the form

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b \qquad v_1, \ldots, v_n, b \in \mathbb{R}^m$$

for unknown scalars $x_1, \ldots, x_n$.

This is:

$$\left(\begin{array}{l} \text{linear combination of} \\ v_1, \ldots, v_n \text{ with unknown weights} \end{array} \right) = (\text{vector})$$

Eg: $x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$

This is equivalent to the system

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(use the column-first definition of the matrix-vector product). But now we're thinking in terms of linear combinations of vectors.

$\rightarrow$ we will draw pictures of these (next time)

Four Ways to Write a System of Eqns:

(1) Linear system

$$\begin{aligned} x_1 - x_2 &= 8 \\ 2x_1 - 2x_2 &= 16 \\ 6x_1 - x_2 &= 3 \end{aligned}$$

(2) Matrix Equation

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(columns)

(3) Augmented Matrix

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

(4) Vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

You still solve a vector equation by putting it into an augmented matrix:

Eg: $x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right]$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right]$$

Solution is $x_1 = -1$, $x_2 = -9$

Important Observation: (!!!!)

$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ has a solution (consistent)

$\iff \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

in which case the solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the vector of weights

In fact, we know $x_1 = -1$, $x_2 = -9$:

$$-1\begin{pmatrix}1\\2\\6\end{pmatrix} - 9\begin{pmatrix}-1\\-2\\-1\end{pmatrix} = \begin{pmatrix}8\\16\\3\end{pmatrix} = \left(\text{linear combination of } \begin{pmatrix}1\\2\\6\end{pmatrix} \text{ \& } \begin{pmatrix}-1\\-2\\-1\end{pmatrix}\right)$$

different b

Eg: $$x_1\begin{pmatrix}1\\2\\6\end{pmatrix} + x_2\begin{pmatrix}-1\\-2\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\\0\end{pmatrix} \rightsquigarrow \left[\begin{array}{cc|c}1 & -1 & 2\\2 & -2 & -2\\6 & -1 & 0\end{array}\right]$$

$$\overset{\text{REF}}{\rightsquigarrow} \left[\begin{array}{cc|c}1 & -1 & 2\\0 & 5 & -12\\0 & 0 & -6\end{array}\right] \quad \text{inconsistent}$$

So $\begin{pmatrix}2\\-2\\0\end{pmatrix}$ is **not** a linear combination of $\begin{pmatrix}1\\2\\6\end{pmatrix}$ & $\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$.

In general:

next time

Column Picture Criterion for Consistency

$Ax = b$ is consistent (has at least one solution)

$\Updownarrow$

b is a linear combination of the columns of A

in which case x = the weights.

We now have 2 ways that linear combinations appear when solving a system of equations:

Linear Systems & Linear Combinations

(1) $Ax=b$ is consistent $\iff$ b is a linear combination of the columns of A.

(2) In this case, the solution set has the form

$$x = \begin{pmatrix}\text{particular} \\ \text{solution}\end{pmatrix} + \begin{pmatrix}\text{all linear combinations} \\ \text{of a set of vectors}\end{pmatrix}$$

Next time: Spans: This is what the set of all linear combinations of a list of vectors looks like.