# Linear Independence

Eg: (HW)

$\text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix},\begin{pmatrix}2\\-5\\1\end{pmatrix},\begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ is a plane.

Why a plane and not $\mathbb{R}^3$? The vectors are coplanar: one is in the span of the others.

$$\frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix}-3\begin{pmatrix}2\\-5\\1\end{pmatrix}=\begin{pmatrix}-1\\5\\12\end{pmatrix}$$ [demo]

Any two non-collinear vectors span a plane:

$$\text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix},\begin{pmatrix}2\\-5\\1\end{pmatrix},\begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}=\text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix},\begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$$

not scalar multiples

This reduces the number of parameters needed to describe this set:

$$x_1\begin{pmatrix}2\\-4\\6\end{pmatrix}+x_2\begin{pmatrix}2\\-5\\1\end{pmatrix}+x_3\begin{pmatrix}-1\\5\\12\end{pmatrix} \quad \text{vs.} \quad x_1\begin{pmatrix}2\\-4\\6\end{pmatrix}+x_2\begin{pmatrix}-1\\5\\12\end{pmatrix}$$

Moreover, the expression with 2 parameters is unique, but with 3 parameters it is redundant:

$$1\begin{pmatrix}2\\-4\\6\end{pmatrix}-1\begin{pmatrix}2\\-5\\1\end{pmatrix}+0\begin{pmatrix}-1\\5\\12\end{pmatrix}=\begin{pmatrix}0\\1\\5\end{pmatrix}=6\begin{pmatrix}2\\-4\\6\end{pmatrix}-7\begin{pmatrix}2\\-5\\1\end{pmatrix}-2\begin{pmatrix}-1\\5\\12\end{pmatrix}$$

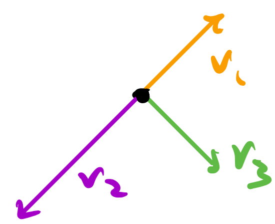
but $\begin{pmatrix}0\\1\\5\end{pmatrix}=x_1\begin{pmatrix}2\\-4\\6\end{pmatrix}+x_2\begin{pmatrix}2\\-5\\1\end{pmatrix}$ only for

[demo] $x_1=1,\ x_2=-1$

We want to formalize this notion that there are "too many" vectors spanning this subspace by saying one is in the span of the others.

In the above example, each vector is in the span of the other 2, but this need not be the case.

Eg: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$




Here $v_2 = -2v_1 + 0v_3$

but $v_3 \notin \operatorname{Span}\{v_1, v_2\}$

We want a condition that means some vector is in the span of the others. Answer: rewrite as a homogeneous vector equation.

Def: A list of vectors $\{v_1, \dots, v_n\}$ is linearly dependent (LD) if the vector equation

$$x_1v_1 + \cdots + x_nv_n = 0$$

has a nontrivial solution. Such a solution is called a linear relation among $\{v_1, \dots, v_n\}$

Eg: $\begin{pmatrix}-1\\5\\12\end{pmatrix} = \overset{\text{move here}}{\phantom{x}} \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix}$

$$\rightsquigarrow 0 = -\begin{pmatrix}-1\\5\\12\end{pmatrix} + \frac{5}{2}\begin{pmatrix}2\\-4\\6\end{pmatrix} - 3\begin{pmatrix}2\\-5\\1\end{pmatrix}$$

is a linear relation

$$\Longrightarrow \left\{\begin{pmatrix}-1\\5\\12\end{pmatrix}, \begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\} \text{ is LD}$$

$$v_2 = \overset{\text{move here}}{\phantom{x}} -2v_1 + 0v_3 \rightsquigarrow 0 = -2v_1 - v_2 + 0v_3$$

is a linear relation

$$\Longrightarrow \{v_1, v_2, v_3\} \text{ is LD}$$

Recall: $Ax=0$ has a nontrivial solution

$\Longleftrightarrow$ $A$ has a free variable

(otherwise the only solution is $x=0$)

$\{v_1, \ldots, v_n\}$ is LD

$\Longleftrightarrow$ $x_1v_1 + \cdots + x_nv_n = 0$ has a nontrivial solution

$\Longleftrightarrow$ the matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ has a free variable

NB: If $x_1v_1 + \cdots + x_nv_n = 0$ and $x_i \neq 0$ then

$$v_i = -\frac{1}{x_i}\left(x_1v_1 + \cdots + x_{i-1}v_{i-1} + x_{i+1}v_{i+1} + \cdots + x_nv_n\right)$$

so $v_i$ is in the span of the others.

LD means some vector is in the span of the others: $x_1v_1+\cdots+x_nv_n=0$ and $x_i \neq 0$ implies $v_i \in \text{Span}\{v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_n\}$

Summary: Let $v_1,\ldots,v_n$ be vectors.
The following are equivalent:

(1) $\{v_1,\ldots,v_n\}$ is linearly dependent

(2) The matrix

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$ has a free variable

(3) Some $v_i$ is in the span of the others

Def: A list of vectors $\{v_1,\ldots,v_n\}$ is linearly independent (LI) if it is not linearly dependent: ie, if the vector equation

$$x_1v_1+\cdots+x_nv_n=0$$

has only the trivial solution.

ie $x_1v_1+\cdots+x_nv_n=0$ implies $x_1=\cdots=x_n=0$.

The logical negation of the Summary above is:

**Summary:** Let $v_1, \dots, v_n$ be vectors.
The following are equivalent:

(1) $\{v_1, \dots, v_n\}$ is linearly independent

(2) The matrix
$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$ does not have a free variable

(3) No $v_i$ is in the span of the others

Roughly, vectors $v_1, \dots, v_n$ are LI if their span is as large as it can be. Every time you add a vector, the span gets bigger!

**Eg:** Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ LI or LD?

In other words, does the vector equation
$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$
have a nontrivial solution?

free ⇒ LD

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{PF}} \begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Take $x_3 = 1$ ⇝ linear relation
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = 0$$

So they're LD [demo]

Eg: Is $\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\-8\\9\end{pmatrix} \right\}$ LI or LD?

In other words, does the vector equation

$$x_1\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} + x_3\begin{pmatrix}7\\-8\\9\end{pmatrix} = 0$$

have a nontrivial solution?

$$\begin{bmatrix}1&4&7\\2&5&-8\\3&6&9\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix}1&4&7\\0&-3&-22\\0&0&32\end{bmatrix}$$

No free variables $\Rightarrow$ only the trivial solution

$\Rightarrow$ these vectors are LI [demo]

Fact: If $\{v_1, \ldots, v_n\}$ is LI and $b \in \mathrm{Span}\{v_1, \ldots, v_n\}$ then there are unique weights $x_1, \ldots, x_n$ such that

$$b = x_1 v_1 + \cdots + x_n v_n$$

In other words, this is not a redundant parameterization of $\mathrm{Span}\{v_1, \ldots, v_n\}$

Proof: Let $A$ be the matrix with cols $v_1, \ldots, v_n$

so $Ax = b \equiv x_1 v_1 + \cdots + x_n v_n = b$

$Ax = b$ is consistent because $b \in \mathrm{Col}(A)$

$\Rightarrow Ax = b$ has one soln because $A$ have no free variables. //

**Linguistic note:** LI, LD are adjectives that apply to a **set of vectors.**

Bad: "A is LI" "$v_1$ is LD on $v_2$ and $v_3$"

Good: "A has LI columns" "$\{v_1, v_2, v_3\}$ is LD"

**Eg:**

- $\{v\}$ is LI $\iff$ $v \neq 0$
- Any set containing the 0 vector is LD: if $v_1 = 0$ then
$$0 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + v_n$$
is a linear relation.
- Suppose $\{v, w\}$ is LD. So there exist $(a,b) \neq (0,0)$ such that $av + bw = 0$.
$$\left.\begin{array}{l} a \neq 0 \rightsquigarrow v = -\frac{b}{a} w \\ b \neq 0 \rightsquigarrow w = -\frac{a}{b} v \end{array}\right\} \text{ v, w are collinear.}$$

$$\boxed{\{v, w\} \text{ is LD} \iff v, w \text{ are collinear.}}$$

- Similarly, $\{u, v, w\}$ is LD $\iff$ $u, v, w$ are coplanar, and so on.
- If $r > n$ then $r$ vectors in $\mathbb{R}^n$ are LD: the matrix $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{bmatrix}$ is wide, so it has a free variable.

  eg. 3 vectors in $\mathbb{R}^2$ are automatically LD.

[demo]

# Basis and Dimension

A basis of a subspace is a minimal set of vectors needed to span (parameterize) that subspace.

**Def:** A set of vectors $\{v_1, \dots, v_n\}$ is a **basis** for a subspace $V$ if:

(1) $V = \mathrm{Span}\{v_1, \dots, v_n\}$

(2) $\{v_1, \dots, v_n\}$ is linearly independent

The **dimension** of $V$ is the number of vectors in any basis. (Fact: all bases have the same size!)

Notation: $\dim(V)$

Spans means you get a parameterization of $V$:

$$b \in V \implies b = x_1 v_1 + \cdots + x_n v_n$$

LI means this parameterization is unique.

**Rephrase:** A spanning set for $V$ is a **basis** if it is linearly independent.

**Eg:** $V = \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$

A basis is $\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \right\}$. (or $\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$)

(1) Spans: because $\begin{pmatrix}-1\\5\\12\end{pmatrix} \in \text{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$

(2) LI: because not collinear.

So $\dim(V) = 2$ (a plane) ✓

Eg: $\{0\} = \text{Span}\{\} \Rightarrow \dim\{0\} = 0$ ✓

Eg: A line $L$ is spanned by one vector

$\Rightarrow \dim(L) = 1$.

In general:

- A point has dimension 0
- A line has dimension 1
- A plane has dimension 2

etc.

Eg: What is a basis for $\mathbb{R}^n$?

The unit coordinate vectors $e_1, \ldots, e_n$.

$n=3$: $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$x_1 e_1 + x_2 e_2 + x_3 e_3 = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

(1) Spans: every vector has this form.

(2) LI: if this $= 0$ then $x_1 = x_2 = x_3 = 0$ ✓

So $\dim(\mathbb{R}^n) = n$ ✓

NB: $\mathbb{R}^n$ has many bases.

eg. $\mathbb{R}^2$ is spanned by any pair of noncollinear vectors: $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$; $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$; $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right\}$, ...

In fact, any nonzero subspace has infinitely many bases! Parameterizations are not unique!

Warning: Be careful to distinguish between these:

| Subspace | Basis | Matrix |
|---|---|---|
| $V = \mathrm{Span}\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ | $\left\{\begin{pmatrix}2\\-4\\6\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}\right\}$ | $\begin{pmatrix}2 & 2\\-4 & -5\\6 & 1\end{pmatrix}$ |

This is a subspace. It is a plane. It has $\infty$ vectors in it.

This is a basis for $V$. It has 2 vectors in it. It is a finite list of data that describes $V$.

This is a matrix $A$. Its columns form a basis for $V = \mathrm{Col}\, A$.

# Bases for Col(A) & Nul(A)

Remember, if someone hands you a subspace, you want to write it as a column space or a null space so you can do computations, like find a basis.

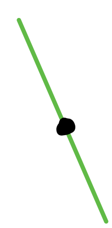
**Thm:** The pivot columns of $A$ form a basis of Col(A).

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{basis: } \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

pivot column

**NB:** Take the pivot columns of the original matrix, Not the RREF. Doing row ops changes the column space!

$$\mathrm{Col}\begin{bmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$



$$\mathrm{Col}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

**Proof:** Let $R$ be the RREF of $A$.

$$A = \begin{bmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the pivot columns are $v_1, v_2, v_4$.

**Note:** $Ax = 0 \iff Rx = 0$ (same solution set)

(1) Spans:
$$\begin{pmatrix}3\\2\\0\\0\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\Rightarrow 0 = -3\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}3\\2\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 0\begin{pmatrix}4\\6\\-1\\0\end{pmatrix}$$

$$\Rightarrow R\begin{bmatrix}-3\\-2\\1\\0\\0\end{bmatrix} = 0 \Rightarrow A\begin{bmatrix}-3\\-2\\1\\0\\0\end{bmatrix} = 0$$

$$\Rightarrow v_3 = 3v_1 + 2v_2$$

$$\boxed{\text{A and R have the same col relations!}}$$

Similarly, $\begin{pmatrix}4\\6\\-1\\0\end{pmatrix} = 4\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 6\begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$$\Rightarrow v_5 = 4v_1 + 6v_2 - v_4$$

Any vector in $\mathrm{Col}(A)$ has the form

$$v = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 + x_5v_5$$
$$= x_1v_1 + x_2v_2 + x_3(3v_1 + 2v_2) + x_4v_4 + x_5(4v_1 + 6v_2 - v_4)$$
$$= (x_1 + 3x_3 + 4x_5)v_1 + (x_2 + 2x_3 + 6x_5)v_2 + (x_4 - x_5)v_4$$

which is in Span$\{v_1, v_2, v_4\}$.

(2) LI: If $x_1v_1 + x_2v_2 + x_4v_4 = 0$ then

$$A\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0 \Rightarrow R\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 = x_4 = 0 \checkmark$$

Consequence: The number of vectors in a basis for Col(A) is equal to the number of pivots of A.

$$\boxed{\text{rank}(A) = \dim \text{Col}(A)}$$

Eg: Find a basis for Span$\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}$

Step 0: Rewrite as Col$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$

Now find pivot columns:

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 2 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis: $\left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \right\}$

2 pivots $\rightsquigarrow$ Span is a plane.

Thm: The vectors attached to the free variables in the parametric vector form of the solution set of $Ax=0$ form a basis for Nul(A)

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{PVF}} x = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

basis: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Proof:

(1) Spans: Every solution $= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ ✓

(2) LI: Think about it in parametric form:

$$\begin{aligned}
0 &= x_1 = -2x_2 + x_4 \\
0 &= x_2 = x_2 \\
0 &= x_3 = -x_4 \\
0 &= x_4 = x_4
\end{aligned}
\Rightarrow x_2 = x_4 = 0 \quad \checkmark$$

↑trivial equations

Consequence:

dim Nul(A) = #free vars = #cols − rank

NB: This is consistent with our provisional definition of the dimension of a solution set as the number of free variables.