MATH 218D-1
PRACTICE FINAL EXAMINATION

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic. You may bring a 8.5×11**-inch note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

Problem 1. [25 points]

Consider the matrix
$$A = \begin{pmatrix} -2/3 & 8/3 & 2/3 \\ 2 & 2 & 3 \end{pmatrix}.$$

a) The rank of A is $r = \boxed{2}$.

b) Explain why it is better to orthogonally diagonalize AA^T instead of A^TA when computing the SVD of A.

Because AA^T is a 2×2 matrix, but A^TA is 3×3.

Here is the matrix AA^T:
$$AA^T = \begin{pmatrix} 8 & 6 \\ 6 & 17 \end{pmatrix}.$$

c) Compute the characteristic polynomial of AA^T.

$$p(\lambda) = \lambda^2 - 25\lambda + 100$$

d) Compute the eigenvalues of AA^T. $\lambda_1 = \boxed{20}$ $\lambda_2 = \boxed{5}$

e) Compute an orthonormal eigenbasis of AA^T. $u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

f) Compute the SVD of A in outer product form.
$$A = 2\sqrt{5}\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}^T + \sqrt{5}\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}^T$$

g) Compute the SVD of A in matrix form: $A = U\Sigma V^T$ for
$$U = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 2\sqrt{5} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{pmatrix} \quad V = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

h) What are the eigenvalues of A^TA?

$20, 5$, and 0

Problem 2. [35 points]

Consider the following matrix and its SVD:

$$A = \begin{pmatrix} -2 & 4 \\ 4 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{30} & 0 \\ 0 & 2\sqrt{5} \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}^T$$

a) Find an orthonormal basis for Col(A).

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

b) Find an orthonormal basis for Nul(A^T).

$$\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

c) Write the SVD of A in outer product form.

$$A = \sqrt{30} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}^T + 2\sqrt{5} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}^T$$

d) Explain why you know $A^T A = \begin{pmatrix} 21 & 3 \\ 3 & 29 \end{pmatrix}$ is positive-definite without doing any calculations.

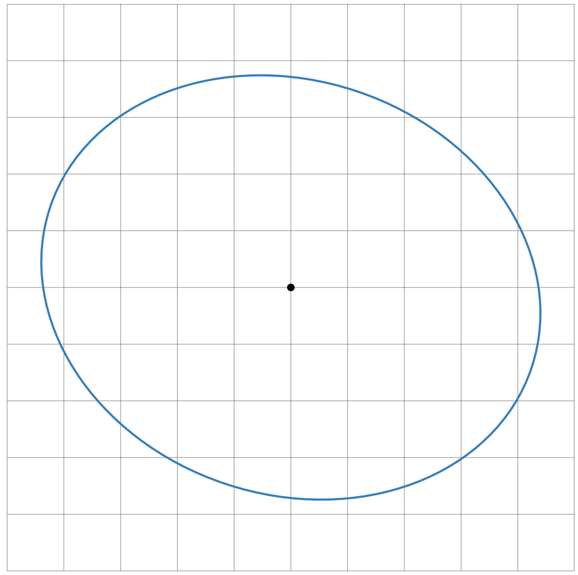
Because A has full column rank.

e) Find an orthogonal diagonalization of $A^T A$: that is, $A^T A = QDQ^T$ for

$$Q = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 30 & 0 \\ 0 & 20 \end{pmatrix}$$

f) Draw the ellipse defined by $q(x_1, x_2) = 21x_1^2 + 29x_2^2 + 6x_1x_2 = 1$. Grid lines are 0.05 units apart.



g) Find the maximum value of $||Ax||^2$ subject to $||x|| = 1$. At which vectors x is the maximum achieved?

max = $\boxed{30}$ $\quad x = \pm\frac{1}{\sqrt{10}}\begin{pmatrix}1\\3\end{pmatrix}$

h) The minimum value of $||Ax||^2$ subject to $||x|| = 1$ is $\boxed{20}$.

Problem 3. [20 points]

Consider the following matrix whose columns consist of 3 data points with 3 measurements each:

$$A_0 = \begin{pmatrix} 4 & 8 & 0 \\ 8 & 1 & 6 \\ 6 & 1 & 2 \end{pmatrix}$$

a) Compute the recentered matrix A (subtract the means of the measurements).

$$A = \begin{pmatrix} 0 & 4 & -4 \\ 3 & -4 & 1 \\ 3 & -2 & -1 \end{pmatrix}$$

The matrix A has SVD

$$A = 3\sqrt{6}\,\frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} + 3\sqrt{2}\,\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$$

and covariance matrix

$$S = \frac{1}{2}AA^T = \begin{pmatrix} 16 & -10 & -2 \\ -10 & 13 & 8 \\ -2 & 8 & 7 \end{pmatrix}.$$

b) The rank of A is $\boxed{2}$.

c) The total variance is $s^2 = \boxed{36}$.

d) The variance of the first measurement is $s_1^2 = \boxed{16}$.

e) The variance in the direction $u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is $s(u)^2 = \boxed{9/2}$.

f) The variance along the plane $V = \text{Span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}^{\perp}$ is $s(V)^2 = \boxed{63/2}$.

g) The line of best fit of the recentered data points is spanned by $u_1 = \boxed{(-2,2,1)}$.

The variance along this line is $s(u_1)^2 = \boxed{27}$, and the error^2 (the variance along the perpendicular plane) is $\boxed{9}$.

h) The direction of second-largest variance is $u_2 = \boxed{(2,1,2)}$, and the variance in this direction is $s(u_2)^2 = \boxed{9}$.

i) Find the plane containing the original 3 data points (the columns of A_0) in the form $p + \text{Span}\{\cdots\}$.

$$\begin{pmatrix}4\\5\\3\end{pmatrix} + \text{Span}\left\{\begin{pmatrix}-2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}\right\}$$

Problem 4. [10 points]

a) Compute the characteristic polynomial of the matrix

$$A_1 = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$p(\lambda) = -\lambda^3 + 4\lambda^2 + 3\lambda - 5$$

b) The matrix

$$A_2 = \begin{pmatrix} 2 & 6 & 6 \\ 3 & -13 & -30 \\ -1 & 6 & 13 \end{pmatrix}$$

has characteristic polynomial

$$p(\lambda) = -(\lambda - 1)(\lambda - 2)(\lambda + 1).$$

Find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

$$C = \begin{pmatrix} -6 & 5 & 2 \\ 3 & -1 & -2 \\ -2 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Problem 5. [15 points]

The matrix

$$A = \begin{pmatrix} -1/2 & 10/3 & 25/3 \\ 7/2 & -31/3 & -85/3 \\ -3/2 & 14/3 & 38/3 \end{pmatrix}$$

has a factorization $A = CDC^{-1}$ for

$$C = \begin{pmatrix} 0 & 5 & 2 \\ 5 & -1 & -2 \\ -2 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

a) Compute C^{-1}.

$$C^{-1} = \begin{pmatrix} 1 & -3 & -8 \\ -1 & 4 & 10 \\ 3 & -10 & -25 \end{pmatrix}$$

b) If $v_0 = (3, 6, -2)$, compute $v_k = A^k v_0$:

$$v_k = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix} + \frac{1}{2^k}\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{3^k}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

c) $\lim_{k\to\infty} A^k v_0 = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$.

Problem 6.

[15 points]

A certain 2×5 matrix A has singular value decomposition

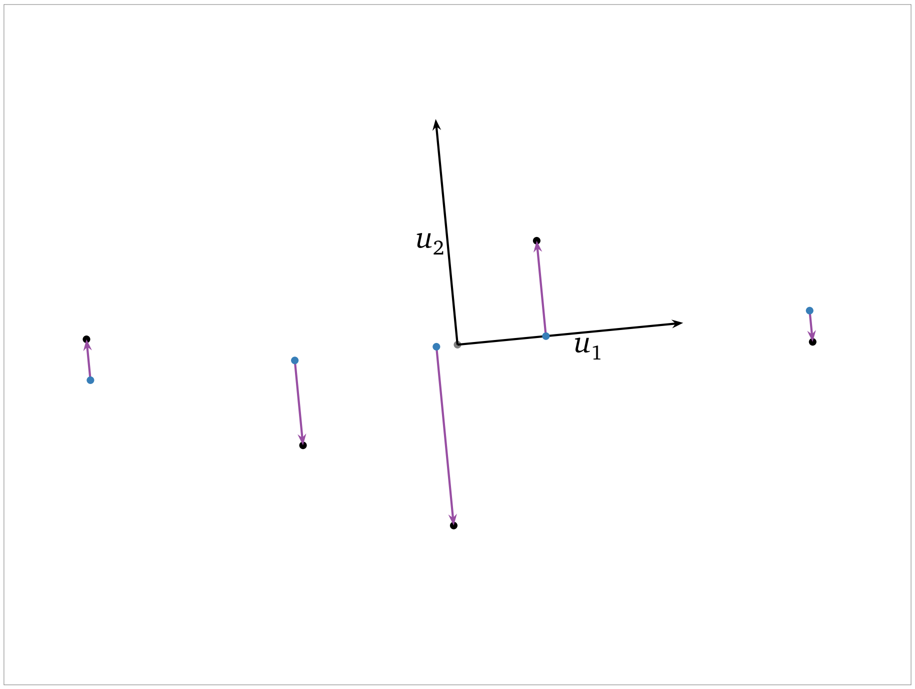
$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T.$$

The columns of A are drawn as dots on the grid below, and the right singular vectors are drawn as arrows.

a) Draw the columns of $\sigma_1 u_1 v_1^T$ as dots on the grid.

b) Draw the columns of $\sigma_2 u_2 v_2^T$ as arrows on the grid.

c) Explain which geometric quantity in the picture corresponds to σ_2^2.
It is the sum of the squares of the lengths of the arrows from **b)**.



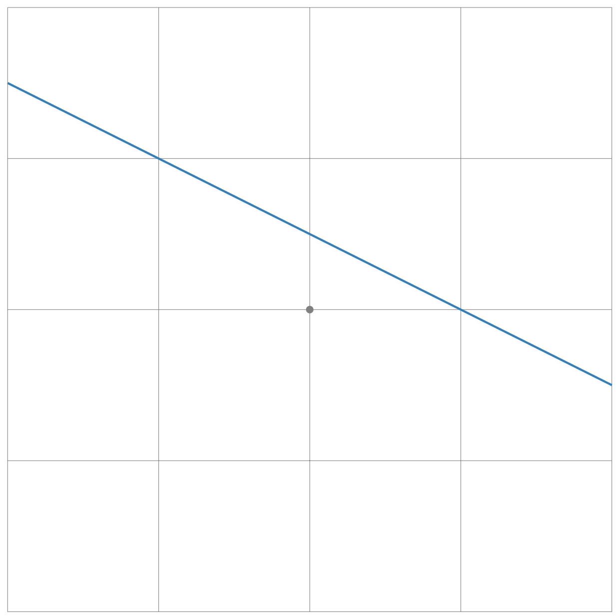
Problem 7. [10 points]

a) Find *all* least-squares solutions of $Ax = b$ in parametric vector form, where

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ -1 & -2 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix}.$$

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

b) Draw a picture of your answer to **a)** in the grid below.



Problem 8.

[30 points]

Short-answer questions: no justification is necessary.

a) Find the LDL^T decomposition of the positive-definite, symmetric matrix

$$A = \begin{pmatrix} 2 & 6 \\ 6 & 21 \end{pmatrix} \rightsquigarrow L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

b) Consider the plane

$$V = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

List three different methods you could use to compute the projection matrix P_V.

Here are a few: (i) The horrible formula. (ii) Compute $P_{V^\perp}$ by projecting onto the line spanned by $(1,2,3)\times(1,1,0)$. (iii) Compute a QR decomposition and use $P_V = QQ^T$.

c) Suppose that A is not diagonalizable and that $A\binom{2}{3} = \binom{4}{6}$. What is the characteristic polynomial of A?

$$p(\lambda) = (\lambda - 2)^2$$

d) Which of the following sets form a basis for the plane

$$V = \{(x, y, z) \in \mathbf{R}^3 \;:\; 3x + 2y + z = 0\}?$$

Fill in the bubbles of all that apply.

○ $\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -6 \end{pmatrix} \right\}$ ● $\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$ ● $\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$

○ $\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ ○ $\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

e) Let A be a 4×4 matrix such that 2 and $1+i$ are eigenvalues of A. Suppose that the 2-eigenspace of A is a plane. Then $\det(A) = \boxed{8}$.

f) Suppose that the solution set of $Ax = \binom{3}{3}$ is the line $\binom{1}{2} + \text{Span}\{\binom{-1}{1}\}$. What is the solution set of $Ax = \binom{-3}{-3}$?

$$-\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

Problem 9.

[20 points]

True/false problems: **circle** the correct answer. No justification is needed.

All matrices in this problem have real entries.

a) **T** (**F**) If V is a subspace of $\mathbf{R}^n$, then $P_V P_{V^\perp} = I_n$.

b) (**T**) **F** If $Ax = b$ has infinitely many solutions for every vector $b \in \mathbf{R}^m$, then A has full row rank.

c) (**T**) **F** The maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$ is the same as the maximum value of $\|A^T y\|^2$ subject to $\|y\| = 1$.

d) (**T**) **F** If σ is a singular value of an invertible matrix A, then σ^{-1} is a singular value of A^{-1}.

e) (**T**) **F** If A has linearly independent columns, then $A^T A$ is positive-semidefinite.

f) (**T**) **F** If $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbf{R}^n$, then $n = 4$.

g) **T** (**F**) Every elementary matrix is diagonalizable.

h) **T** (**F**) A diagonalizable 5×5 matrix has 5 distinct eigenvalues.

i) (**T**) **F** If A is an $n \times n$ matrix with linearly dependent rows, then the columns of A do not span $\mathbf{R}^n$.

j) **T** (**F**) The diagonal entries of a square matrix are its eigenvalues.