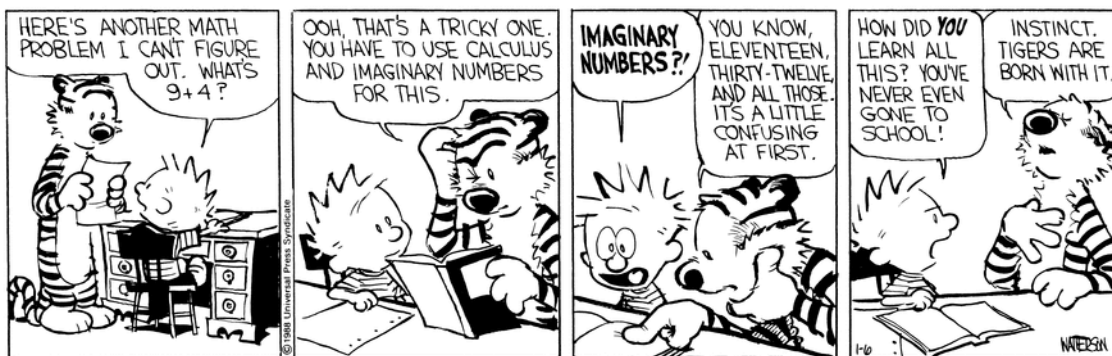


MATH 218D-1
MIDTERM EXAMINATION 2

Name		Duke NetID	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **simple calculator** for doing arithmetic, but you should not need one. You may bring a **3 × 5-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



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Problem 1.

[20 points]

Consider the following matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 3 & -1 & -2 & -1 \\ 5 & 1 & -6 & -7 \\ 1 & -3 & 2 & 5 \\ -2 & 2 & 0 & -2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let V be the subspace $\text{Nul}(A)$.

- a) V is a (fill in one bubble) point line plane space in \mathbf{R}^{\square} .

b) Find a basis for V .

$$\left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\}$$

c) Find a basis for V^\perp .

$$\left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\}$$

d) Compute the orthogonal projection b_V for $b = (0, 1, -1, 1)$.

$$b_V = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 1]

[Scratch work for Problem 1]

Problem 2.

[15 points]

In this problem we will find the best fit plane

$$z = B + Cx + Dy$$

through the data points

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}.$$

a) Find the matrix A and the vector b such that the coefficients (B, C, D) of the best-fit

plane is the least-squares solution of the matrix equation $A \begin{pmatrix} B \\ C \\ D \end{pmatrix} = b$.

$$A = \begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix} \quad b = \begin{pmatrix} \\ \\ \end{pmatrix}$$

b) Compute the QR decomposition of A .

$$Q = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \quad R = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

[Scratch work for Problem 2]

(Problem 2, continued)

c) Find the best-fit plane, using your answer to b) or otherwise.

$$z = \boxed{} + \boxed{}x + \boxed{}y$$

d) Find the orthogonal projection of the vector b onto $V = \text{Col}(A)$.

$$b_V = \begin{pmatrix} \\ \\ \end{pmatrix}$$

[Scratch work for Problem 2]

Problem 3.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & -4 & 8 \\ 0 & 3 & -4 \\ 0 & 2 & -3 \end{pmatrix}.$$

a) Compute the characteristic polynomial of A .

$$p(\lambda) =$$

b) Compute a basis for the 1-eigenspace of A .

$$\left\{ \begin{array}{l} \\ \\ \end{array} \right\}$$

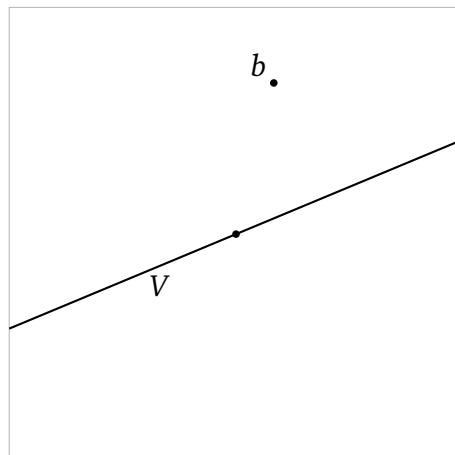
c) Is 2 an eigenvalue of A ? Why or why not?

[Scratch work for Problem 3]

Problem 4.

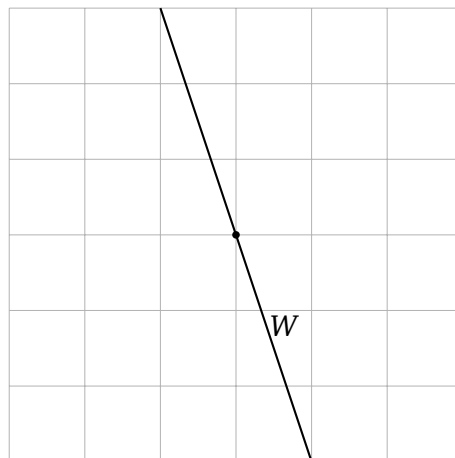
[10 points]

A subspace V and a vector b are drawn below.



- Draw and label the vectors b_V and b_{V^\perp} .
- Draw and label any eigenvector of the projection matrix P_V .

A different subspace W is drawn below. Grid marks are one unit apart.



- Compute the projection matrix P_W .

$$P_W = \begin{pmatrix} & \\ & \end{pmatrix}$$

[Scratch work for Problem 4]

Problem 5.

[20 points]

Short-answer questions: no justification is necessary.

a) $\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}^{15} \begin{pmatrix} 2 \\ -1 \end{pmatrix} =$

b) Suppose that V is a subspace of \mathbf{R}^n . Which of the following statements are *equivalent* to “ $\dim(V) = n$ ”? Fill in the bubbles of all that apply.

$\det(P_V) \neq 0$

$V = \mathbf{R}^n$

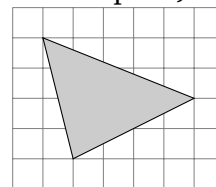
1 is an eigenvalue of P_V

$P_{V^\perp} = 0$

$\text{rank}(P_V) = n$

$b_V = b$ for every $b \in V$

c) Compute the area of the triangle in the picture. (Grid marks are one unit apart.)



area =

d) Find a basis for the orthogonal complement of

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} \right\}.$$

$\left\{ \begin{pmatrix} \\ \\ \end{pmatrix} \right\}$

e) Which **one** of the following statements is correct? Fill in the appropriate bubble.

An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ .

An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ .

An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v .

An eigenvector of A is a vector v such that $Av = \lambda v$ for a scalar λ .

[Scratch work for Problem 5]

Problem 6.

[20 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write “no way, man,” or use your favorite colloquialism instead. You need not justify your answers.

a) A matrix Q with orthonormal columns such that $P_V = QQ^T$, where V is the plane

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} \right\}.$$

b) An invertible matrix A with characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$.

c) A non-invertible matrix A such that $A^T A$ is invertible.

d) A 2×2 projection matrix of rank 1.

[Scratch work for Problem 6]